SHAPE & SIZE
AREAS, VOLUMES, CENTROID POSITIONS
Mensuration formulas and heuristic principles

Introduction

In elementary mathematics we often require a formula for the area or volume of, or the position of the centroid in, a standard figure or solid. (The position of the centroid completes the picture by telling us how the area or volume is distributed.) The formula has a shape dictated by a set of principles which we may call *heuristic* because, unlike axioms, which remain in the background, they help us derive or check the expression. If methods are the *tactics* we employ, these principles suggest the *strategies*. The distinction is not always clear-cut. I have for example included among my principles the principle of moments, which Archimedes himself famously called 'The Method', and Cavalieri's principle. I illustrate each principle with a few examples. Invariably we invoke the principles in combination, which is why the same formulas appear under many headings. The table at the end keys the main entries. (Columns A and B should in fact have an entry for every shape.) Where other headings are relevant in a particular case, we have also written the reference in **bold**.

The methods required are elementary: Euclidean geometry, especially similar triangles, the geometry of simple affine transformations, and arguments which depend on infinitesimals but do not use the mechanics of integral calculus, namely Cavalieri's principle and the centroid theorems of Pappus. The latter deals with the many figures with central symmetry we shall meet.

It will be useful to look at these topics first:

Archimedes' 'method': all of **I**
Cavalieri's principle: all of **D**
The Pappus centroid theorems: **J.9**

The principles we have chosen to isolate are:

(A) The requirement for dimensional consistency
(B) The requirement for algebraic symmetry
(C) The relabelling principle
(D) Cavalieri's principle
(E) The principle of the limiting case
(F) The principle of continuity
(G) The principle of similarity
(H) The dissection principle
(I) The principle of moments
(J) Analogy
(K) The principle of maximum symmetry
(L) The complete information principle
(M) The principle of assuming the minimum

(A) The requirement for dimensional consistency

In terms of length, the degree of any expression for area must be 2, for volume, 3. If *p*, *q*, *r* are lengths, pq^2 is a volume, as is $\frac{p^4r^2}{q^3}$, but not pq^2r.

1) Check the numerator in this expression for the volume of the frustum of a right circular cone with base radius *R*, top radius *r* and height *h*: $\frac{\pi(R^2+Rr+r^2)h}{3}$.

2), **3)**, **4)**, ... Check all the formulas which follow.

As an aside,
it can be useful to derive dimensionless numbers because they allow us to examine shape independent of size. The isometric quotient $Q_A = \frac{4\pi A}{L^2}$ for a closed curve of length *L*, area *A*, takes its greatest value 1 for a circle. The smaller the number, the spikier the shape. You may check that the analogous expression for a closed three-dimensional surface of area *A*, volume V is $Q_V = \frac{36\pi V^2}{A^3}$.

(B) The requirement for algebraic symmetry

If, in a particular case, the linear dimensions a and b play the same role, then $a^2 + ab + b^2$ is a plausible area formula, $2a^2 + ab + b^2$ not.

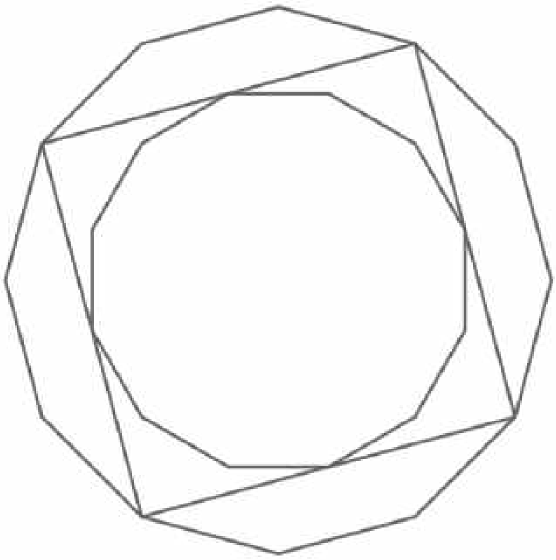
1) Check the formula above (**A.1**) for the volume of the cone frustum, where r and R play the same role.

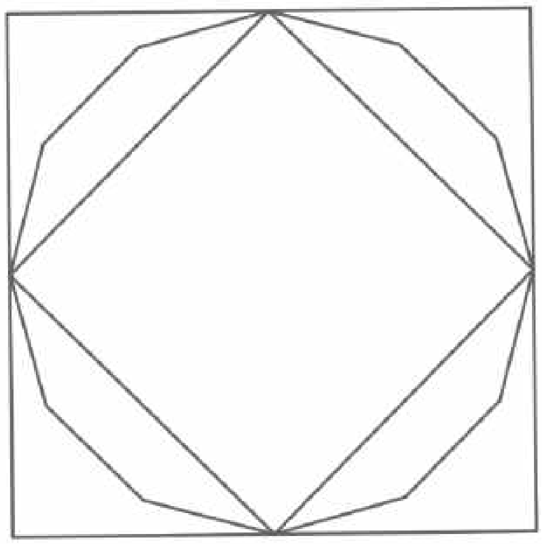
2) Check the formula below (**D**.**2**) for the volume of the pyramid frustum, where a and b play the same role.

3) By de Gua's theorem (**J.4**), the area of a triangle with side lengths $\sqrt{p^2 + q^2}$, $\sqrt{q^2 + r^2}$, $\sqrt{r^2 + p^2}$ is $\frac{\sqrt{p^2q^2+q^2r^2+r^2p^2}}{2}$. Since the set of lengths is symmetrical in *p*, *q*, *r*, so must the expression for area be. Compare the Heron formula (**E.1).**

By the commutative law, *ab* is identical to *ba*. This fact gives us an economic way of solving the following problem.

4) The figure on the left shows how a regular 12-gon has been inscribed in a square, which has in turn been inscribed in a second regular 12-gon. We are asked what fraction of the outer 12-gon is occupied by the inner one. Let *f* be the fraction of the square occupied by the inner 12-gon; *g* the fraction of the outer 12-gon occupied by the square. By dissection we find that $f = \frac{3}{4}$, $g = \frac{2}{3}$. The required fraction is $fg = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$. But it is also *gf*. The figure on the right makes this transposition geometrically, where a single dissection provides the answer.

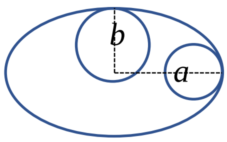




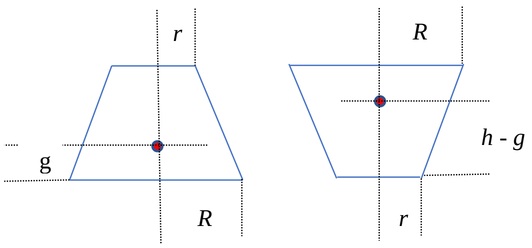
(C) The relabelling principle

Another way of stating **B** is that we may swap *a* for *b* without changing the formula. But we may also do this in cases where the formula is not symmetrical in *a* and *b* but their roles are equivalent in some sense.

1) The Cartesian equation of the ellipse is symmetrical in *a* and *b*, the lengths of the semi-major and semi-minor axes respectively. These axes are distinguished by length but are equivalent geometrically: they are both symmetry axes. The radius of curvature at the ends of the major axis is $\frac{b^2}{a}$. What is the radius of curvature at the ends of the minor axis? We can imagine we've simply relabelled the two axes, thereby swapping *a* and *b*. The answer must therefore be $\frac{a^2}{b}$. Notice incidentally the **dimensional consistency**. In the **limiting case** $a = b$ and we have the radius of a circle.



2) For the cone frustum, the height of the centroid, g, above the base is $\frac{(R^2+2Rr+3r^2)h}{4(R^2+Rr+r^2)}$, an expression which is not symmetrical in *R* and *r*. If we swap *r* and *R*, we reverse their roles. This relabelling has a simple geometrical interpretation: we've turned the frustum upside down:



The expression must then be $\frac{[(Denominator)-(R^2+2Rr+3r^2)]h}{(Denominator)} = \frac{(3R^2+2Rr+r^2)h}{4(R^2+Rr+r^2)}$, swapping the coefficients in the numerator, as required.

3) In **J.6** we use as the base of a right triangle, first a leg, then the hypotenuse. This enables us to equates two different expressions for the area. And we do the analogous thing in three dimensions for the volume of a right tetrahedron.

(D) Cavalieri's principle

Modern integral calculus deals in rectangles of infinitesimal width, sheets of infinitesimal thickness. It assumes a hypothetical plasticine which is infinitely malleable, completely incompressible and infinitely divisible. And it enables us to calculate without comparison with a model. But, if we have that model, we can invoke an older use of infinitesimal quantities, Cavalieri's principle. In two dimensions, it concerns two closed curves between the same parallel tangents. If they have the same length of section along every parallel between the tangents, they have the same area, (and, by extension, their centroids lie on the same parallel). In three dimensions, it concerns two solids between the same parallel tangent planes. If two solids have the same cross-sectional area in every parallel plane between the tangent planes, they have the same volume, (and by extension, their centroids lie in the same plane). By a further extension, we can argue that a constant multiplying factor multiplies the dimension (area or volume in the respective case) by the same factor.

1) In the case of the pyramid and cone, these cross-sectional areas are directly proportional to height above the base. The volume of a pyramid with base area *a* and height *h* is $\frac{ah}{3}$. By Cavalieri's principle this must also be true for a cone. Indeed we can regard both as examples of the general cone.

2) Compare the volume formulas for (**I**), our cone frustum, and **(II)**, the frustum of a square pyramid with base edge *a* and top edge *b*, when their volumes are equal:

(I) $\frac{\pi(R^2+Rr+r^2)h}{3} = \frac{h}{3}(\pi R^2 + \pi Rr + \pi r^2)$,
(II) $\frac{h}{3}(a^2 + ab + b^2) = \frac{h}{3}\left[\left(\sqrt{\pi}R\right)^2 + \left(\sqrt{\pi}R\right)\left(\sqrt{\pi}r\right) + \left(\sqrt{\pi}r\right)^2\right]$.

At each height we have swapped a circle for a square of the same area.

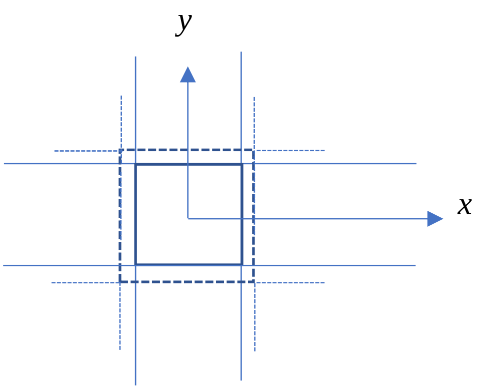
3) For a right circular cone, the fractional height of the centroid above the base is ¼. This must also be true for a square-based pyramid, a tetrahedron, or any other general cone.

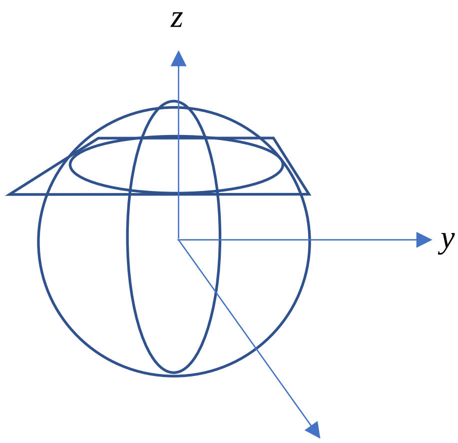
4) Consider the intersection of two like cylinders, of radius *r*, whose axes cut at right angles. What is the volume of the common solid? The answer was known to Archimedes and to the ancient Chinese.

Set up coordinate axes, one cylinder axis lying along the the *x*-axis, the other along the *y*-axis.

Sections parallel to the *x*-*y* plane are squares of varying size.

Sections parallel to the *x*-*z* and *y*-*z* planes are equal circles of radius *r*



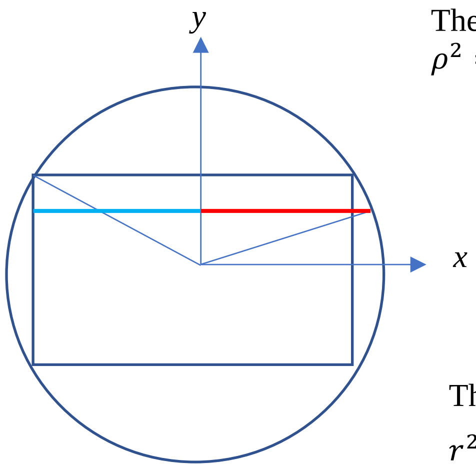


x

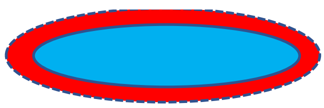
We can now invoke our corollary to Cavalieri's principle. If the sections parallel to the *x*-*y* plane were circles, the solid would be a sphere of volume $\frac{4\pi r^3}{3}$. But, for each such circle, we must substitute a square of edge $2r$, that is, we must multiply the volume of the sphere by $\frac{4}{\pi}$, resulting in a final volume of $\frac{4}{\pi} \times \frac{4\pi r^3}{3} = \frac{16r^3}{3}$, that is $\frac{2}{3}$ the volume of a circumscribing cube.

5) We analyse the 'napkin ring' problem in the same way we treat a famous result of Archimedes at **I.1**. The surprising solution prompts an entry in section **L**.

To produce the ring, we bore a cylindrical hole of radius *r* through a sphere of radius *R*, resulting in a ring of height h. What is the volume of material in the ring?

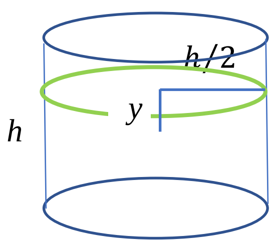


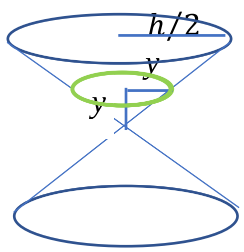
The radius ρ of the red circle is given by
$\rho^2 = R^2 - y^2$.



The radius r of the blue circle is given by
$r^2 = R^2 - \left(\frac{h}{2}\right)^2$.

Subtracting, we have the area of the annulus, $\pi(\rho^2 - r^2) = \pi\left[\left(\frac{h}{2}\right)^2 - y^2\right]$.
This gives us different inner and outer radii for an annulus of the same area. These annuli have a fixed outer radius and an inner radius which varies linearly with height.
As we move up the vertical section, we complete a cylinder of height h, radius $\frac{h}{2}$, from which we must subtract a double cone of the same height and radius.

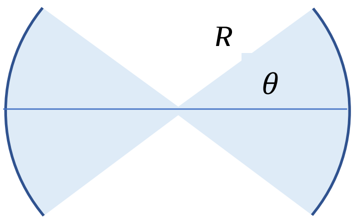


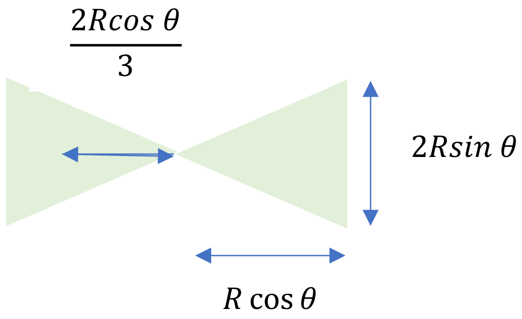


$$\pi\left(\frac{h}{2}\right)^2 h - 2\frac{\pi}{3}\left(\frac{h}{2}\right)^2\left(\frac{h}{2}\right)$$
$$= \frac{4}{3}\pi\left(\frac{h}{2}\right)^3.$$

This is the volume of a sphere whose diameter is the height of the ring.

Here is another way to arrive at the same result. At **H.7** we invoke Archimedes' 'hatbox' theorem to show that the volume of the blue solid is $\frac{4\pi R^3}{3}\sin\theta$. To obtain our napkin ring we must subtract the green solid.





The volume of this is generated by rotating a green triangle about the vertical axis. At **J.7** we learn that the result is:

the area of the triangle × the circumference of the circle its centroid travels, that is:

$$\frac{1}{2} R\ cos\ \theta \times 2R \sin\theta \times 2\pi \frac{2R\cos\theta}{3}$$
$$= \frac{4\pi R^3}{3} \sin\theta[1-(\sin\theta)^2]\ .$$

Subtracting the green from the blue, we have $\frac{4}{3}\pi(R\sin\theta)^3$, as required.

(E) The principle of the limiting case

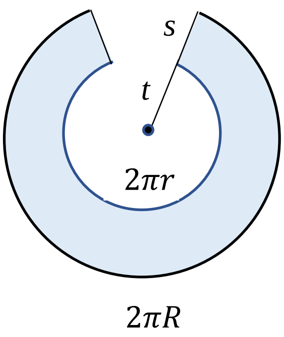
A shape can be viewed as one of an infinite series obtained by continuously varying a parameter. This series is bounded by limiting cases, where the shape acquires a new name but the same formula describes the quantity being measured. The domain of every function we consider is the set of non-negative real numbers. Of particular interest is what happens to the value of the function when the value of a particular variable is zero.

1) The area of a cyclic quadrilateral with sides *a*, *b*, *c*, *d*, semi-perimeter *s*, is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$. If we deform the quadrilateral till the side of length *d* vanishes, we have a triangle, whose area must therefore be $\sqrt{s(s-a)(s-b)(s-c)}$.

2) The side length *c* of a triangle, whose other sides, lengths a and b, contain an angle θ, is given by $c^2 = a^2 + b^2 - 2ab\cos\theta$. If we deform the triangle till θ becomes a right angle, the third term on the right vanishes and we have $a^2 + b^2 = c^2$.

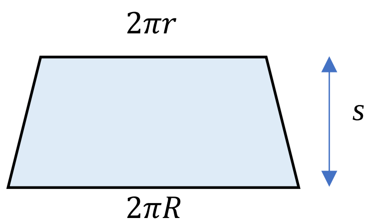
3) If the slant height of our cone frustum is *s*, the area of the sloping part is $\pi(R+r)s$.

We can show this from the net, which looks like this:



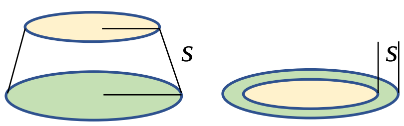
Our area is a fraction $\frac{2\pi r}{2\pi t}$ (or $\frac{2\pi R}{2\pi(s+t)}$) of the complete annulus, area $\pi[(s+t)^2 - t^2]$.
We eliminate *t* from the ratio $\frac{s+t}{t} = \frac{R}{r}$ to obtain our result.

Alternatively, dividing the annulus sector into concentric strips of infinitesimal thickness, and stretching them out, We have this trapezium:



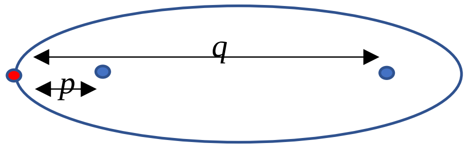
(Compare **J.3**). This illustrates the principle that a simple result may betray a simple way of arriving at it.

If we think of the frustum as made of foam and squash it flat (i.e. project it onto the base), it becomes an annulus where *s* is the width of the ring, $R - r$, giving an area of $\pi(R+r)(R-r) = \pi(R^2\text{-}\,r^2)$.

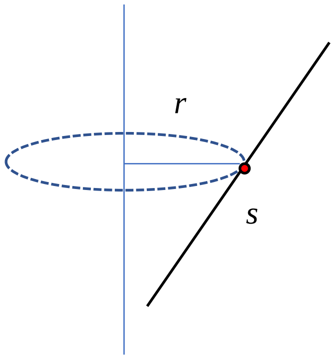


If *r* becomes equal to *R*, we have a cylinder of which *s* is the height, giving an area for the curved surface of $2\pi Rs$. If *r* vanishes, we have a cone, with area πRs. Our original volume formula for the cone frustum simplifies to $\frac{\pi R^2 h}{3}$.

4) We learn that the sum of the distances of a point on an ellipse from its two foci is constant. What is that distance? We have only to choose a point at the end of the major axis to see that the answer, $p + q$, is $2a$, where a is the length of the semi-major axis:



5) Here is some line segment of length s whose midpoint is distant r from an axis of rotation. After one revolution it generates a hyperboloid of one sheet.



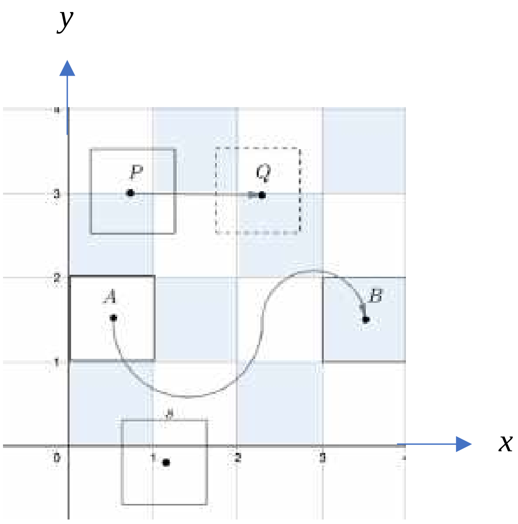
At **J.9** we meet the Pappus centroid theorems. One tells us that the area swept out is $2\pi rs$.

In the limiting case, the segment cuts the axis and we have a cone. The radius of the base is then $R = 2r$ and we have the formula πRs for its surface area.

(F) The principle of continuity

Given the base, the area of a triangle is a continuous function of its height. We assume this 'algebraic' continuity in all the cases we deal with in these notes. 'Geometric' continuity often follows from the algebraic version.

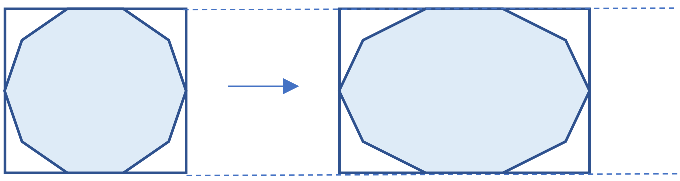
For example, we have a square, *s*, which we can translate over this checkerboard.



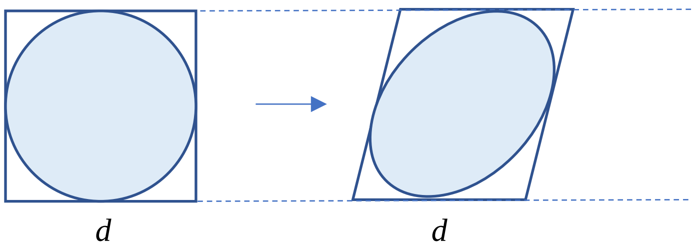
We move *s* by some route from *A* to *B*. Let the fraction of the square occupied by blue be $f(x)$. We see that $f\left(\frac{1}{2}\right) = 0, f\left(3\frac{1}{2}\right) = 1$. The intermediate value theorem says that, in this interval, the value of $f(x)$ must pass through every value between 0 and 1.

We start from *P* and end at *Q*. Let the fraction of blue be $g(x)$. If $g\left(\frac{2}{3}\right) = \frac{1}{2}$, and $g\left(3\frac{1}{3}\right)$ also $= \frac{1}{2}$, there must be some route between *P* and *Q* along which $g(x) = \frac{1}{2}$ for every *x* value. (We show a simple one.)

The adjective 'continuous' applied to the transformations we use in these notes can be defined in terms of ratio.
We use an affine transformation, the stretch, at **H.2**, **H.4**. This preserves ratios of area and ratios of volume.



Another affine transformation is the shear, which preserves areas, volumes and their ratios. For example, here the ellipse on the right has the same area as the original circle:

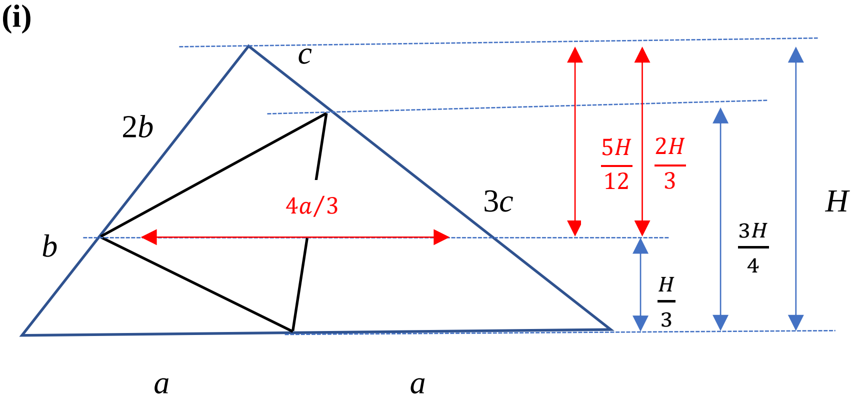


These properties gives us two ways to solve the following problem.

1) Find the fraction of the outer triangle occupied by the inner one.

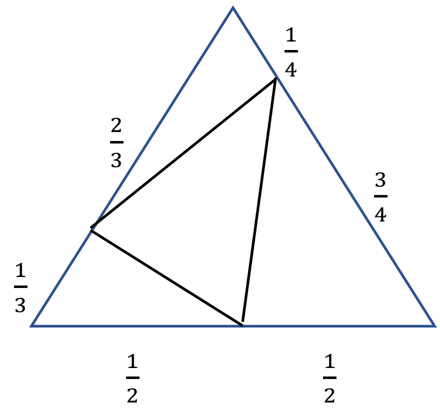
In both approaches we subtract the three small outer triangles from the main one.

(i)



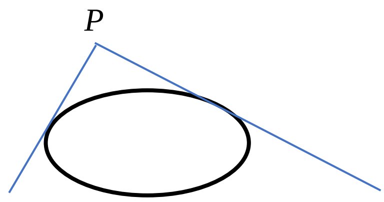
On the left we've managed to work out all the relative height and base measurements we need to carry out the plan. But the ad hoc nature of this approach may lead to mistakes.

(ii) On the right, by some combination of stretches and shears, we've transformed the outer triangle into a convenient special case, an equilateral triangle, and given it unit side. Since all the outer angles are 60°, we can use the area formula 'half the side product times the sine of the angle between'. We have a factor $\frac{\sqrt{3}}{4}$ which cancels, leading to the fraction $\frac{24-4-4-9}{24} = \frac{7}{24}$.



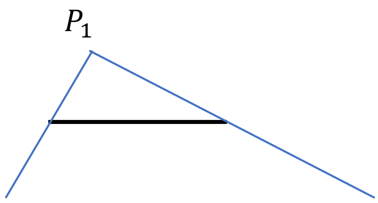
Topological continuity is the assertion that all neighbouring points which start together end together. This is too vague in the metric context we are working in here. Because of this vagueness, a result we derive using continuity must be established more rigorously. Here is a case in point.

2) We are asked for the locus of the point P, in which two perpendicular tangents to an ellipse with semi-major axis a, semi-minor axis b, meet:

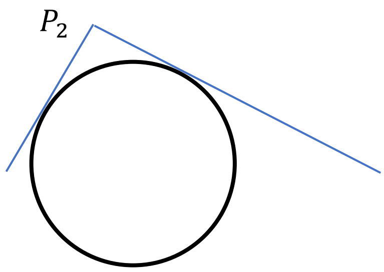


We take limiting cases:

$b = 0$



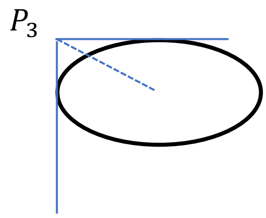
$a = b$



P_1 follows a circle by the converse of Thales' theorem.

P_2 follows a circle by the symmetry of the circle.

We take a special case:

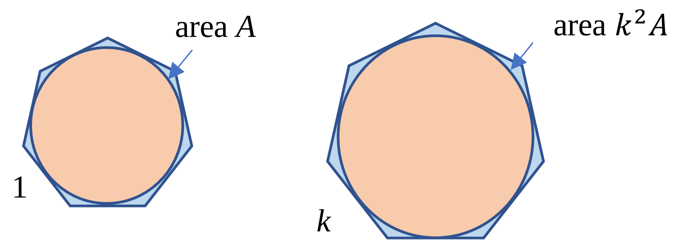


If the locus is a circle, P_3 reveals that it has a radius of $\sqrt{a^2 + b^2}$, and the figures for P_1, P_2 are consistent with this. But, for all this circumstantial evidence, we would need to do the necessary coordinate geometry to confirm that the locus is indeed a circle.

(G) The principle of similarity

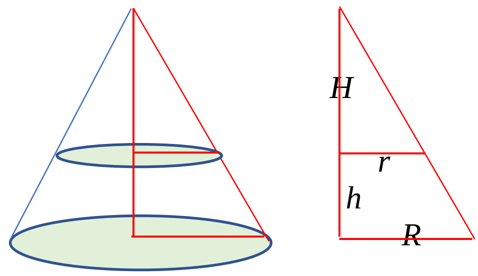
The same ratio holds between any two corresponding linear elements in two similar figures; the square of that ratio holds for the areas of any two corresponding 2-dimensional features; the cube for the volumes of any two 3-dimensional features.

1) Here is how it applies in two dimensions:



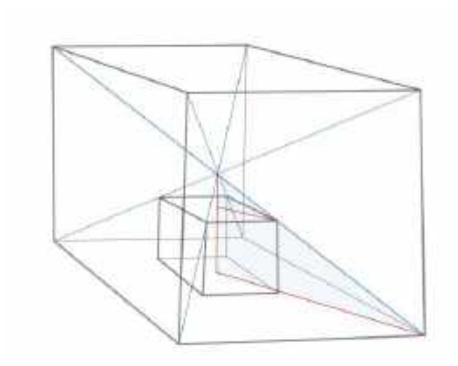
2) We use similarity when we derive our cone frustum volume formula by subtracting a small cone from a larger, similar one.

We eliminate H by similar triangles:
$\frac{H+h}{H} = \frac{R}{r}$.
The smaller cone has volume
$\frac{\pi r^2 H}{3}$;
the larger cone, $\frac{\pi R^2(H+h)}{3}$.
Substituting and simplifying, we have our result.

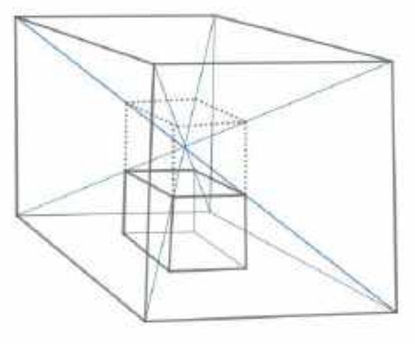


3) A small cube is placed within a larger, unit, cube so that 4 vertices lie in a face and 4 on space diagonals of the larger cube. What is the edge of the smaller cube?

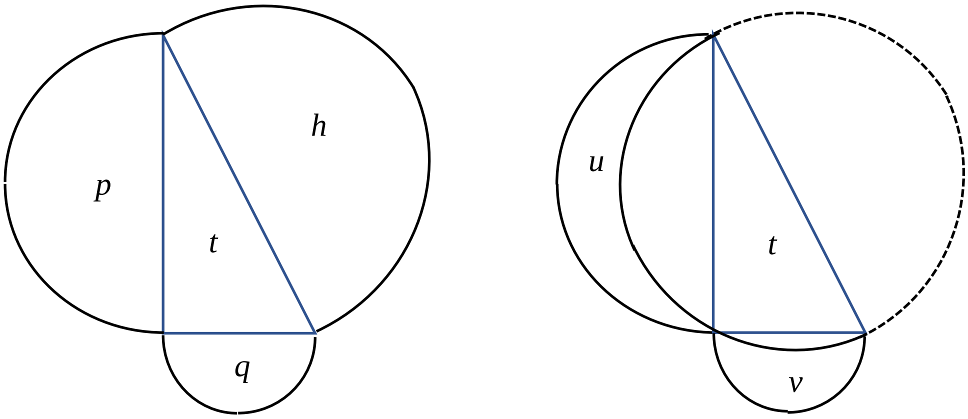
Working with the similar right triangles outlined in red, after some algebra and work with surds, we find the answer: 1/3.



Alternatively, we imagine sitting a like cube on top of the original. By similarity, it shares space diagonals and a centre with the larger cube. By symmetry, therefore, 3 like cubes have the height of the larger one.

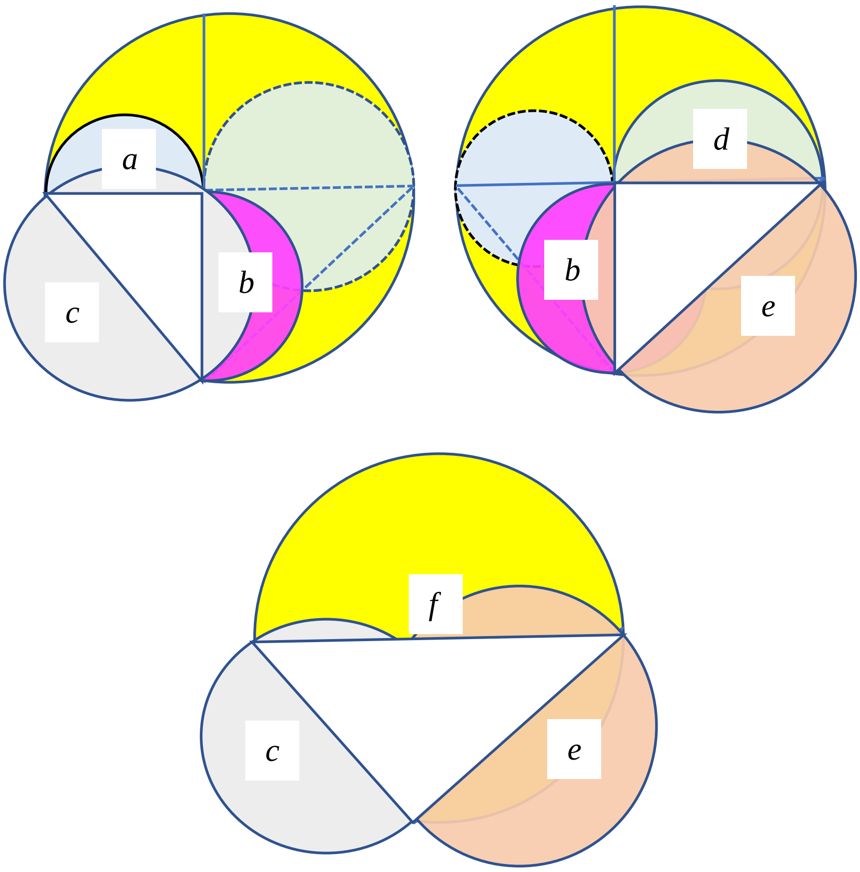


4) We can substitute any mutually similar figures for the squares on the sides of a right triangle and still satisfy Pythagoras' theorem, for example semicircles. If we reflect the semicircle in the hypotenuse and remove common regions, we have Alhazen's result that the 'lunes' are equal in combined area to the right triangle. (See also **7**.)



$$
\begin{aligned}
p+q &= h\\
p+q+t &= h+t\\
p+q+t-h &= t\\
u+v &= t.
\end{aligned}
$$

5) Roger B. Nelsen uses the left-hand figure above to prove a famous result of Archimedes. We have a kite (not drawn) in a circle. We take one of the main constituent right triangles and split it into two further right triangles (white). We equate the semi-circle sum for each and for the whole triangle.



$$
\begin{aligned}
a+b &= c.\\
b+d &= e.
\end{aligned}
$$

$$a+2b+d=c+e=f.$$

$$2b=f-a-d.$$

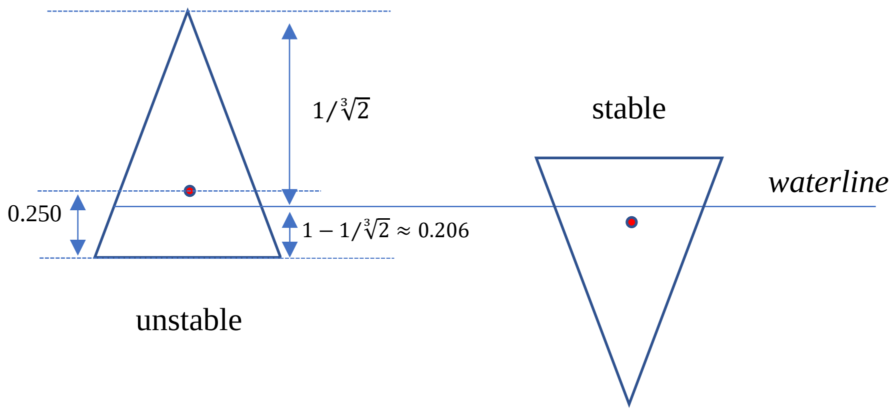
$2b$ is the area of the magenta circle, which has half the vertical diagonal of the kite as diameter.

$f-a-d$ is the area of the upper yellow region, which Archimedes called an *arbelos* (shoe-maker's knife).

So the area of the arbelos is that of the circle with half the tangent to the two smaller circles as diameter.

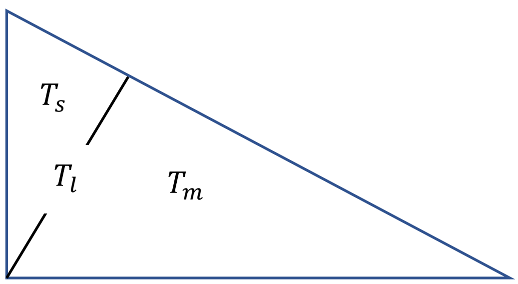
6) A wooden tetrahedron of relative density ½ floats in water. Does it float with an apex pointing up or an apex pointing down?

As noted above, a tetrahedron is a general cone. The only property we need is that all cones with a given apical angle are similar and their centroids lie ¼ way above the base. Let the cone have unit height. It displaces half its volume of water. This represents a cone whose height is $\frac{1}{\sqrt[3]{2}}$. Were it to float with an apex pointing up, we would have this situation:



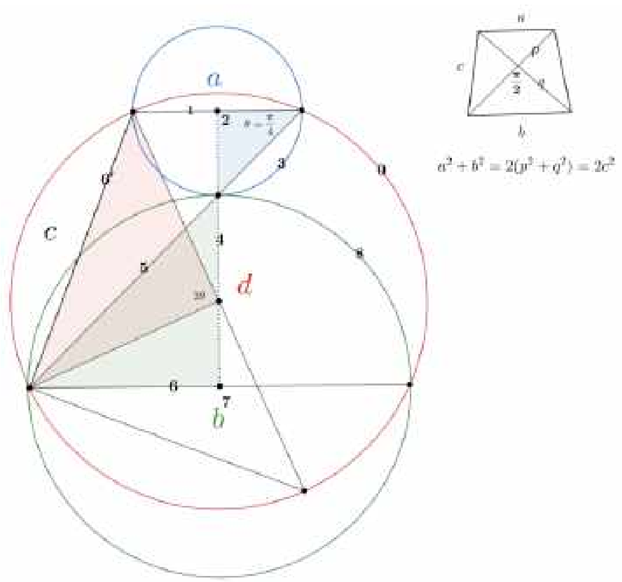
This would be unstable since any displacement would result in a turning moment about the centre of buoyancy, the centroid of the displaced body of water, which lies beneath the waterline. It must therefore float with the apex pointing down.

7) Arguably, the most economical proof of Pythagoras' theorem is Einstein's. Since the only requirement of the figures on the sides of the triangles is that they be similar, Einstein chooses the right triangle itself, T_l.



The smaller triangles share their hypotenuses with the sides of T_l and are flipped inwards. Thus we have T_s on the short side, T_m on the intermediate side, making up between them T_l itself on the hypotenuse.

8) In this example we wish to compare the areas of the blue, green and red circles. To do so, we only need take the blue, green and red isosceles right triangles, which, being the same fraction of their parent circles, deputise for them.



The construction stages are numbered.

0 The red circle
1 A chord
2 Its centre
3 The blue circle
4 The line of centres of the red and blue circles

5 A line from one end of the chord through the cut of the blue circle and the line of centres, defining the blue isosceles right triangle
6 A chord parallel to the original one from the cut of **5** with the red circle
6' A chord joining the left hand ends of the first two chords
7 The centre of the second chord, (which lies on the red-blue line of centres), defining the green isosceles right triangle
8 The green circle

The inset figure shows the isosceles trapezium defined by the first two chords. We see by completing isosceles right triangles on the diameters of the blue and green circles that the diagonals of this trapezium are perpendicular. It follows that $a^2 + b^2 = 2c^2$.

We return to the main figure. Since $\theta = \frac{\pi}{4}$, the angle at the centre of the red circle $= 2\theta = \frac{\pi}{2}$. This gives us the red isosceles right triangle. Duplicating this, we have a similar triangle with hypotenuse d, where $d^2 = 2c^2 = a^2 + b^2$.

So the area of the red circle is the sum of that of the blue circle and the green circle.

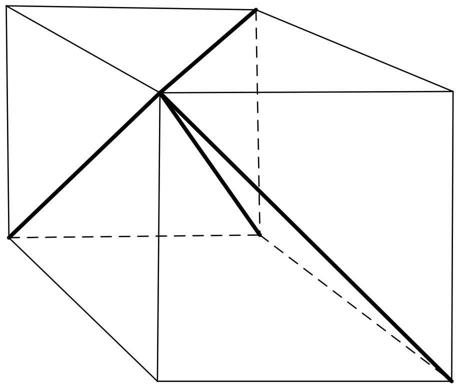
We can check this result by considering the **limiting case**. When $a = 0$, the green and red circles coincide; and, relabelling, when $b = 0$, the blue and red circles coincide.

(H) The dissection principle

Anything we can measure is conserved on dissection, whether length, area, volume or angle.

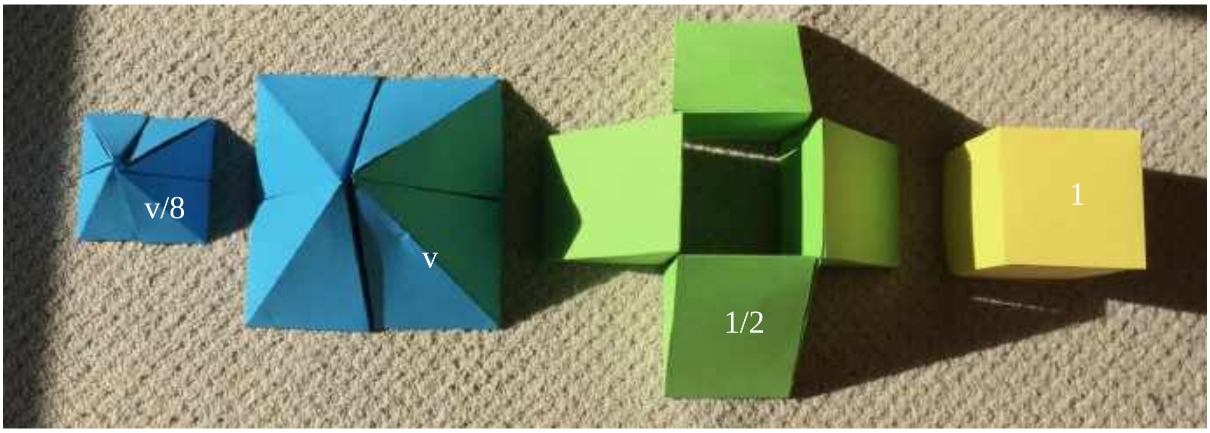
We learn a lot from alternative dissections of the same shape. (Many proofs of Pythagoras' theorem exploit this possibility.) Here is an example where we invoke **similarity**.

1.) We shall call a pyramid on a square base with apex vertically above a base vertex, a 'corner' pyramid. We know we can dissect a cube into 3 congruent corner pyramids, whose height is equal to a base edge length:

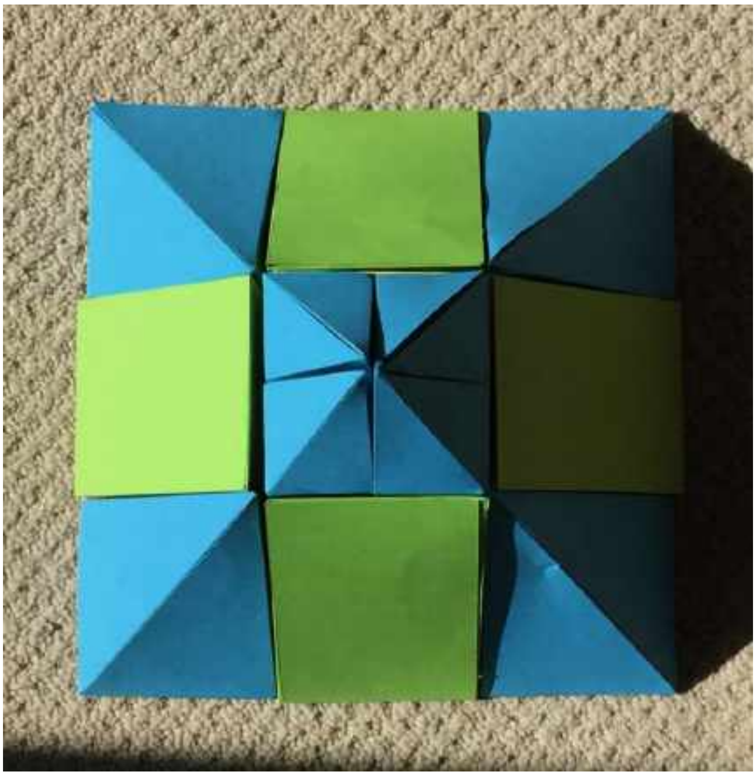


So the volume of that special pyramid is 1/3 that of the containing prism (a cube). For a corner pyramid of general height we can apply a one-way stretch normal to the base and argue that volume relations are preserved. Thus we establish that the volume of a general corner pyramid is 1/3 that of the containing prism. But, as in the special case, we can show this purely by dissection.

We build a pyramid from these pieces, whose volumes are shown relative to the yellow block, which would be the containing prism for our corner pyramid:



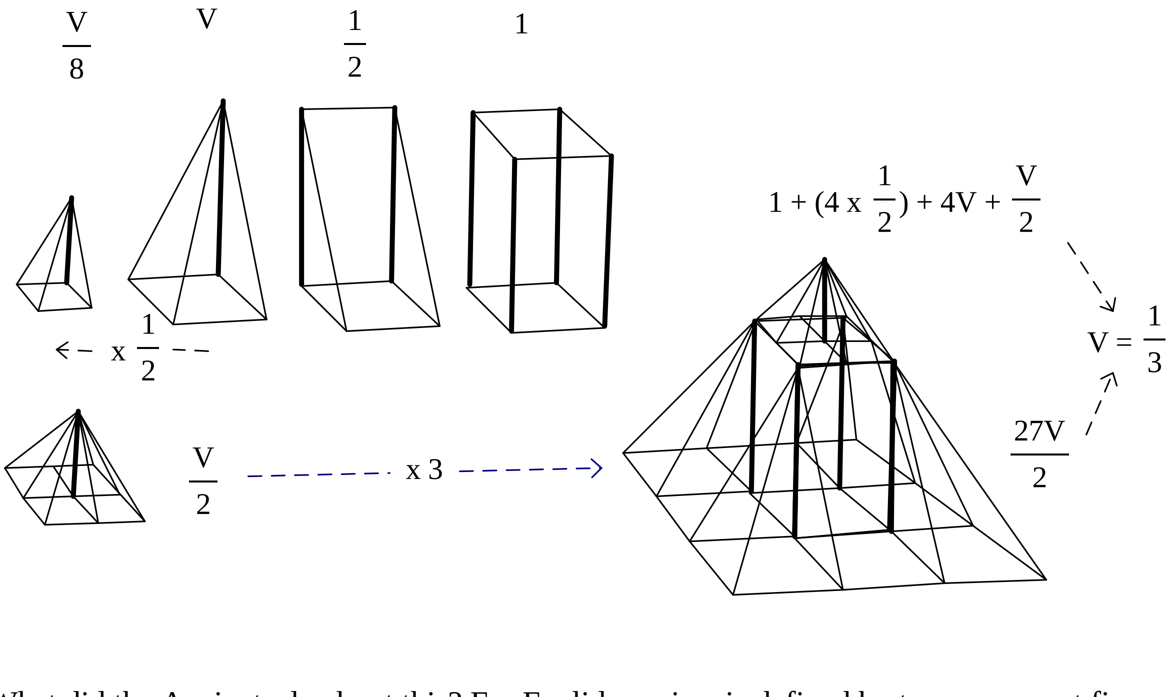
To work out the volume of the big pyramid, we can either add the bits, or treat it as a pyramid made from the 4 small blue pieces scaled 3^3 by volume.



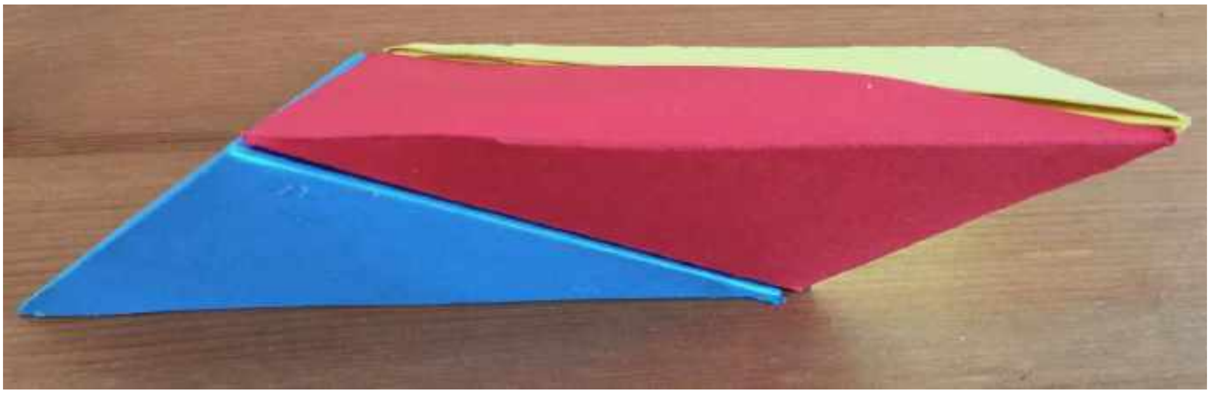
This gives us an equation we can solve for v .

We see below how the algebra works out.

As you'll realise, the result, $v = \frac{1}{3}$, applies not only to the corner pyramid in its containing prism but also to the right pyramid made from 4 such units in *its* containing prism.



What did the Ancients do about this? For Euclid, a prism is defined by two congruent figures lying in parallel planes. So the side faces are parallelograms, not necessarily rectangles. A pyramid is defined by a point in one plane and a figure in a parallel plane. So again we can have 'skew' versions. When you realise Book XII, Proposition 7, as a dissection, you get this:



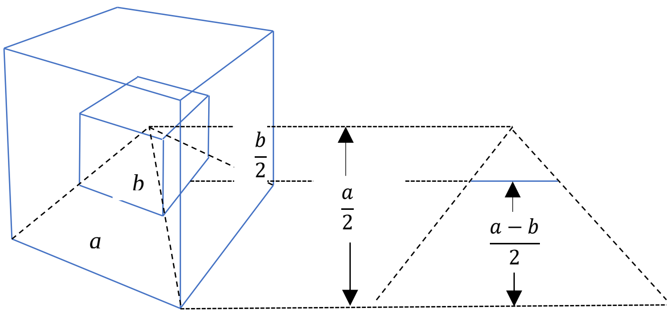
The parallel faces are to left and right, a blue triangle to the left and a yellow triangle to the right. Thus the blue and yellow pieces have congruent bases and the same height (the distance between the parallel faces), and therefore the same volume. Now note that the red and blue triangles facing you are congruent. Turn the model over on to that parallelogram as base. You will find that the red and blue pieces have the same height, and, having the same base, the same volume. So the three tetrahedra have equal volumes, each therefore 1/3 of its containing prism. (And Euclid uses this proposition to establish the same result for the square-based pyramids we are concerned with.)

It is easier to make models for the special case of a right prism with equilateral end faces. This results in two congruent tetrahedra (the blue and the yellow) and faces as follows:

	Faces		
Tetrahedra	Equilateral	Right	Isosceles
Blue, Yellow	1	2	1
Red	0	2	2

(You find you can assemble the pieces as prisms (right or skew) with each of the three constituent triangles as end faces, which is rather nice.)

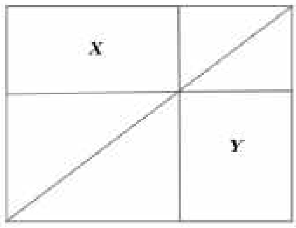
2) The Ancient Babylonians obtained the frustum of a pyramid by removing a cube of edge b from the centre of a cube of edge a, yielding 6 congruent pieces, each of volume $\frac{a^3-b^3}{6}$:



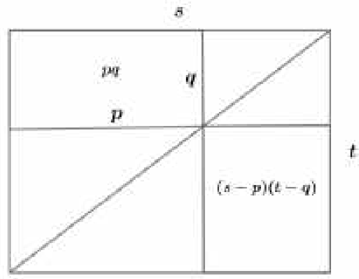
If, in modern terms, we perform a vertical one-way stretch on the frustum so that the height changes from $\frac{a-b}{2}$ to h, increasing the volume in proportion to this, we have the general volume formula quoted above:

$$\frac{h}{(a-b)/2} \times \frac{a^3-b^3}{6} = \frac{2h}{a-b} \times \frac{(a-b)(a^2+ab+b^2)}{6} = \frac{h}{3}(a^2+ab+b^2).$$

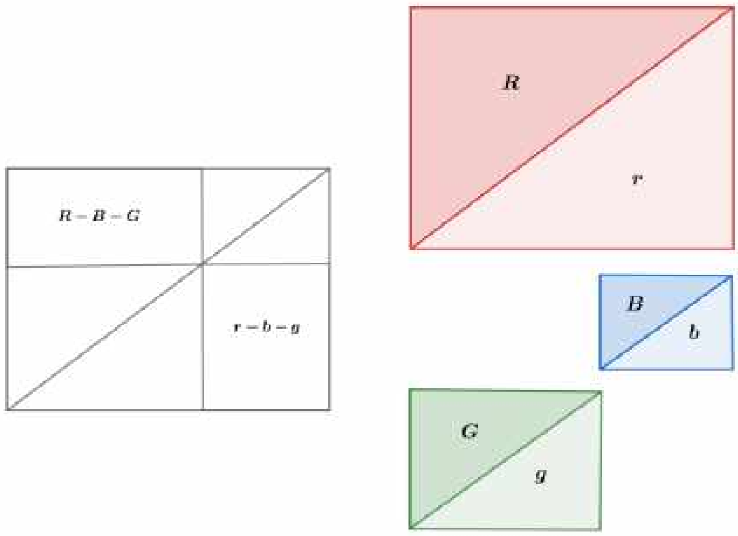
3) We are asked to show that the rectangles X, Y are of equal area.



We can identify similar triangles and derive the result from these ratios: $\frac{t-q}{p} = \frac{q}{s-p}$.

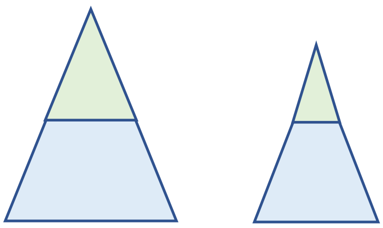


But a dissection argument avoids any algebra:

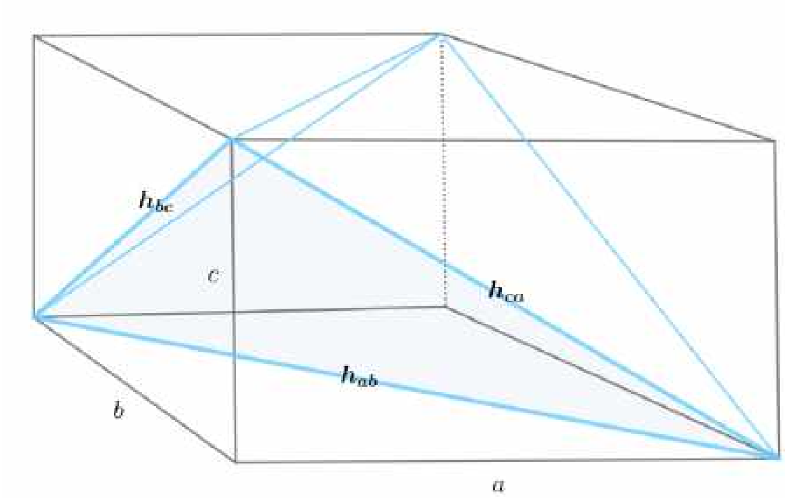


We can think of the whole figure as a ‘proof without words’, or we can construct the proof like this:
1. Bring together two congruent right triangles *R*, *r* as shown.
2. Choose a point on their common hypotenuse and cut at right angles to the legs.
This creates two pairs of congruent right triangles (*B*, *b*; *G*, *g*).
3. Discard these.
4. Examine the two rectangles which remain.
Since we began with figures of equal area and removed figures of equal area, we must be left with figures of equal area.

It is important to realise however that, except in one special case, the rectangles are not similar. Metrically, similarity involves a ratio, which involves multiplication/division, but here we are concerned with addition/subtraction. The distinction is clear in the case of the trapezia within this pair of similar isosceles triangles:

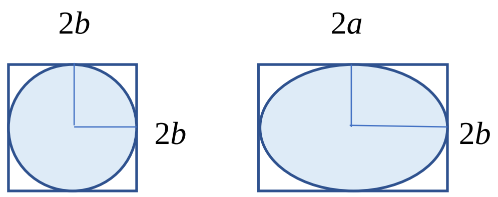


4) This tetrahedron is inscribed in a cuboid of volume *abc*.



Removing the four right tetrahedra, each of volume $\frac{abc}{6}$, we are left with the volume of the tetrahedron, $\frac{abc}{3}$.
The four faces are congruent, therefore of equal area. Taking the **limiting case** of the tetrahedron in a cube, the faces are equilateral and the tetrahedron regular therefore. In the unit cube, its volume is $\frac{1}{3}$ and each tetrahedron edge of length $\sqrt{2}$. To find the volume of a regular tetrahedron of unit edge, we must observe the principle of **similarity** and scale this volume by $\left(\frac{1}{\sqrt{2}}\right)^3$, giving us a final volume of $\frac{\sqrt{2}}{12}$.

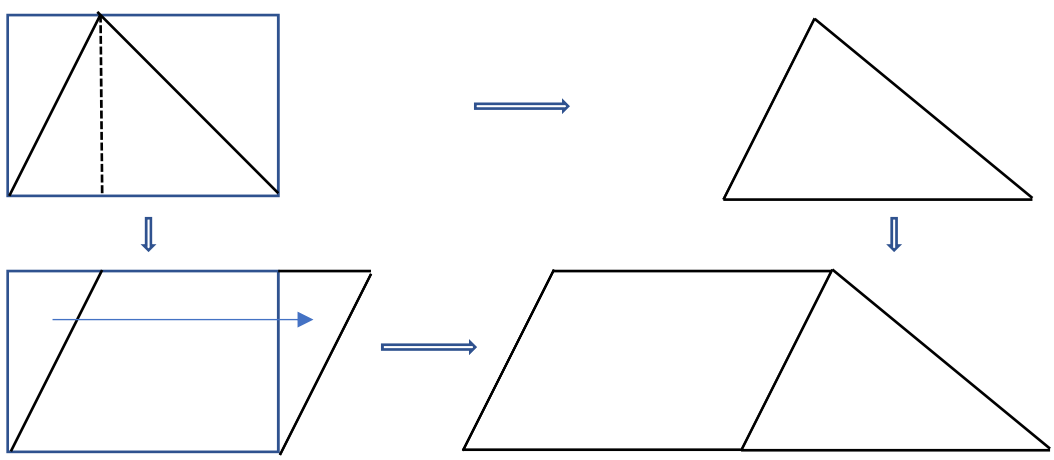
We can think of applying stretches parallel to the edges of the unit cube to produce the cuboid (or vice versa). Such a transformation preserves ratios of volume (and, in two dimensions, ratios of area). Thus, starting with a circle in a square of edge $2b$, we have the area of an ellipse with axes $2a$ and $2b$:



Area: πb^2 Area: πab

And, by **analogy**, beginning with a sphere in a cube of edge $2b$, with volume $\frac{4\pi b^3}{3}$, the volume of an ellipsoid with axes $2a, 2b, 2c$ is $\frac{4\pi abc}{3}$.

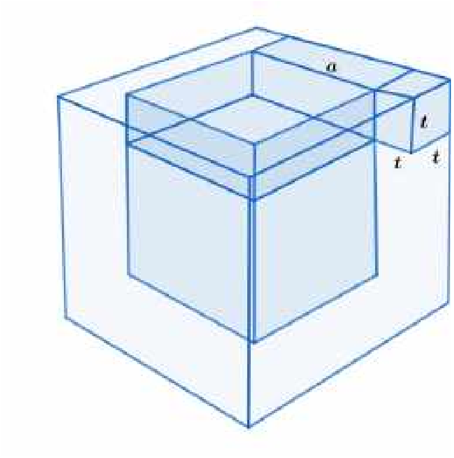
5) The simplest use of dissection to determine shape areas is summarised in this scheme:



In the following case, expanding an algebraic expression enacts a dissection.

6) What is the volume, *V*, of the shell enclosing the central cube?

We use the identity $p^3 - q^3 = (p - q)(p^2 + pq + q^2)$.

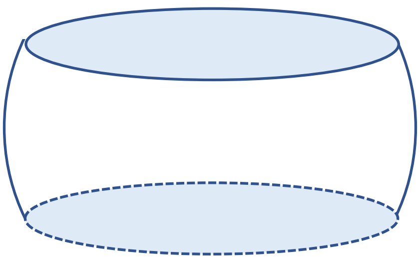


V
$= (a + 2t)^3 - a^3$
$= [(a + 2t) - a][(a + 2t)^2 + a(a + 2t) + a^2]$

$= 6(a^2t) + 12(at^2) + 8(t^3)$.

Magically, the algebra has told us that there are 6 face prisms, 12 edge prisms, 8 corner cubes, and given us the volume of each.

7) This solid of revolution is a spherical segment symmetrical about a great circle.
The picture shows a sink plug this shape.
With a great circle vertical, it lets the water out;
with a great circle horizontal, it keeps it in.
We shall dissect it in two different ways.

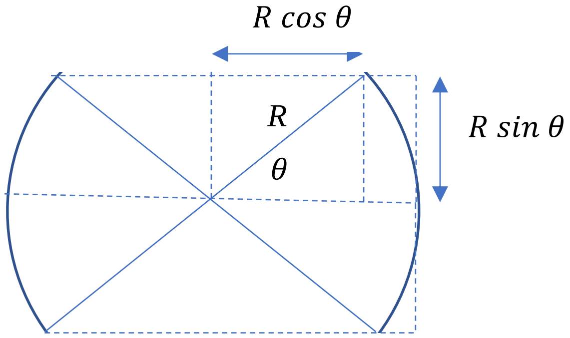


(i) The first way, from the centre of the sphere we construct two equal cones. This leaves a solid of revolution which is a spherical sector. (**J.2** shows the **analogy** with the two-dimensional case. A spherical sector can be of any shape at the surface of the sphere. As shown at **J.2**, when considering its volume, we can invoke **Cavalieri's principle** and treat it as a cone with that base, and apex at the sphere's centre.) By Archimedes' 'hatbox' theorem, the surface area is directly proportional to $\sin\theta$. And, since the volume is directly proportional to the surface area, the volume simplifies to $\frac{4\pi R^3}{3}\sin\theta$. Adding the two cones, each of volume $\frac{\pi(R\cos\theta)^2 R\sin\theta}{3} = \frac{\pi R^3(\cos\theta)^2\sin\theta}{3} = \frac{\pi R^3\sin\theta[1-(\sin\theta)^2]}{3}$, we have a total of $\frac{2\pi R^3\sin\theta[3-(\sin\theta)^2]}{3}$.
We can see that, as θ approaches $\pi/2$, the cone pair vanishes and we are left with the formula for a complete sphere. As θ approaches 0, the double cone fills the outer cylinder until, in the limit, it occupies exactly one third, and the other solid two thirds.

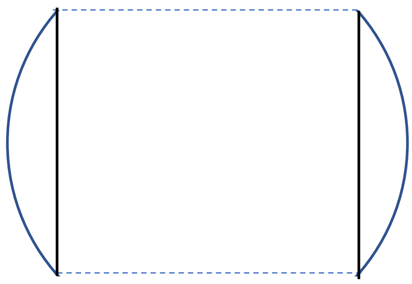
(ii) We split the whole solid into a central cylinder and a 'napkin ring'. We find at **D.5** that the volume of the ring is that of a sphere with the same height. We can rearrange the expression above to show this.

$$\frac{2\pi R^3 \sin\theta[3-(\sin\theta)^2]}{3} = \pi(R\cos\theta)^2 \times 2R\sin\theta + \frac{4\pi}{3}(R\sin\theta)^3.$$

(i)



(ii)



(I) The principle of moments

Archimedes famously borrowed the law of the lever from statics - a science he had himself just created! - to calculate a number of areas and volumes.

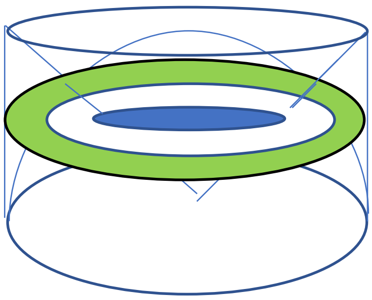
We shall derive one of his classic results and use that in turn to find the position of the centroid of a hemisphere.

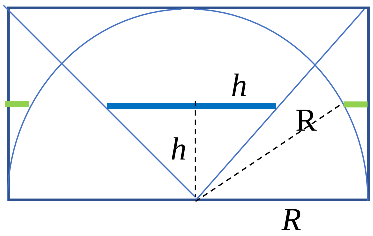
1) Archimedes hung spheres, cones and a cylinder from the arms of an imaginary balance in such a way that, when you 'scan' across the assembly, you find everything in perfect balance, just as you find the slices equal in area as you 'climb' a pair of **Cavalieri** solids. The process enabled him to use the fact that, in the figure below, the green area is equal to the blue area. We can confirm this by Pythagoras' theorem:

Blue area = πh^2.   Green area = $\pi R^2 - \pi(R^2 - h^2) = \pi h^2$.

The upshot is that the volume of a cylinder equals the sum of those of an inscribed cone and hemisphere and furthermore they stand in the ratio

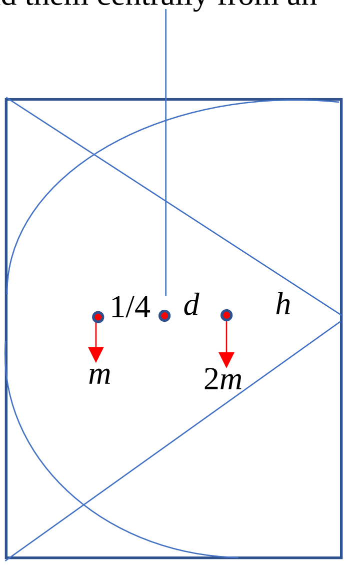
Cylinder : Hemisphere : Cone :: 3 : 2 : 1.





2) We know that the fractional height of the cylinder's centroid above its base is ½ by symmetry, that of the cone, ¼ (**J.5**). What is it for the hemisphere?

As we have seen, the inter-penetrating cone and hemisphere are equivalent to a cylinder, so we know the centroid is at the half-way point and can suspend them centrally from an imaginary balance:



(The arrows indicate weights but these are proportional to the masses.)

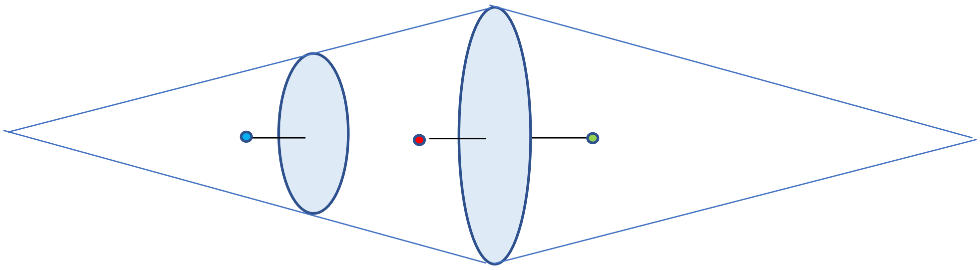
A distance ¼ left of centre hangs a mass *m* (the cone).

A distance *d* right of centre hangs a mass 2*m* (the hemisphere).

Taking moments, $d.2m = \frac{1}{4}$.m .

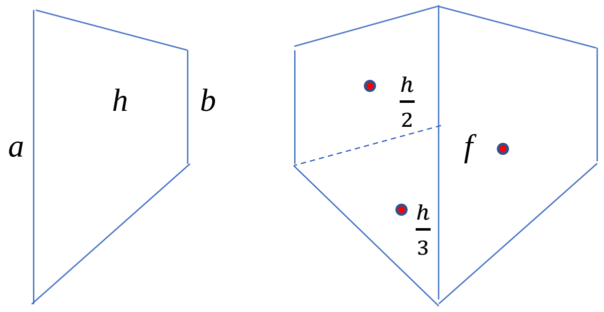
Therefore *d* = 1/8 and the fractional height of the hemisphere's centroid above its base, *h*, is $\frac{1}{2}$ - $\frac{1}{8} = \frac{3}{8}$.

3) Here is how we might go about finding g above, the height of the centroid of a cone frustum above the base, without the machinery of integral calculus (though, with it, the calculation is much simpler). Having found the height of the cone centroid as $\frac{h}{4}$, using the principle of **similarity**, the **dissection** principle and the principle of **moments**, and balancing solids like this ...



... we find we have all the information we need to label the figure and perform the calculation.

4) We can find the height, f, of the centroid of a trapezoidal lamina above the base from this figure:



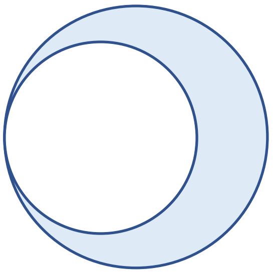
The result is $\frac{(a+2b)h}{3(a+b)}$.

This expression has essentially the same form as that for the height of a cone frustum centroid above the base, and **relabelling** here has the same significance.
The **limiting case** $a = b$ gives $\frac{h}{2}$, as expected.

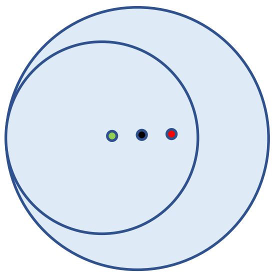
In the next examples we are removing from a figure a **similar** one.

5) This crescentic lamina has half the area of the complete disk. Where is its centroid (the red point)?

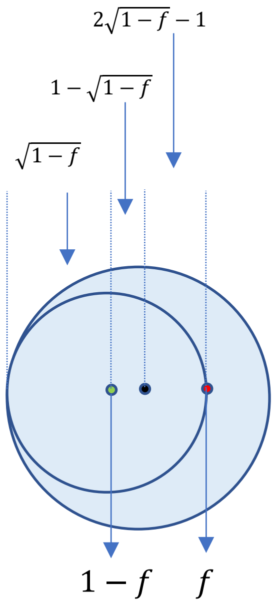
Let the outer disk have unit radius. The disk removed must then have radius $\frac{\sqrt{2}}{2}$.



We restore the disk. Its centre is the green point.
Taking moments about the centre of the whole disk (black),
the black-red distance must equal the black-green distance, which is
$1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$.



A more interesting result was remarked by Paul Glaister. What fraction f of the area of the whole disk must the crescent be that its centroid lies on the inner arc?



This time the situation is as shown on the left.
The figure shows what we need for our moments equation.
f is the solution to this equation:

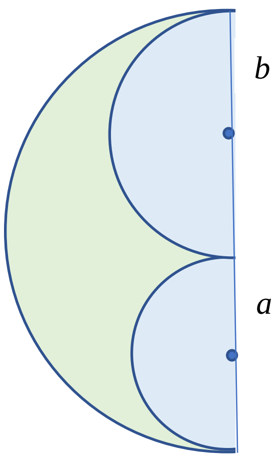
$$f(f^2+f-1)=0.$$

Since we require $0<f<1$, the solution is

$\frac{(\sqrt{5})-1}{2}$, the 'small' golden ratio, τ.

In a similar way we find that the centroid of a cone of height 4 from which a similar cone of half the volume has been removed is distant $\frac{2^{\frac{2}{3}}-2^{\frac{1}{3}}}{2}$ from the centroid of the whole cone; and that the centroid of a hemisphere of height $\frac{8}{3}$ from which a hemisphere of half the volume has been removed is likewise $\frac{2^{\frac{2}{3}}-2^{\frac{1}{3}}}{2}$ from the centroid of the whole hemisphere.

6) In section **G** we treat the arbelos. It's interesting to find the distance of the centroid from the diameter of the outer circle. We need a result from **J.9**. The distance of the centroid of a semicircle of radius *r* from the diameter is $\frac{4r}{3\pi}$. Write $\frac{4}{3\pi}=t$.



As at **J.9** we balance the whole semicircle against the component parts: the two small semicircles and the arbelos (green), whose centroid lies at a distance *x* from the diameter.

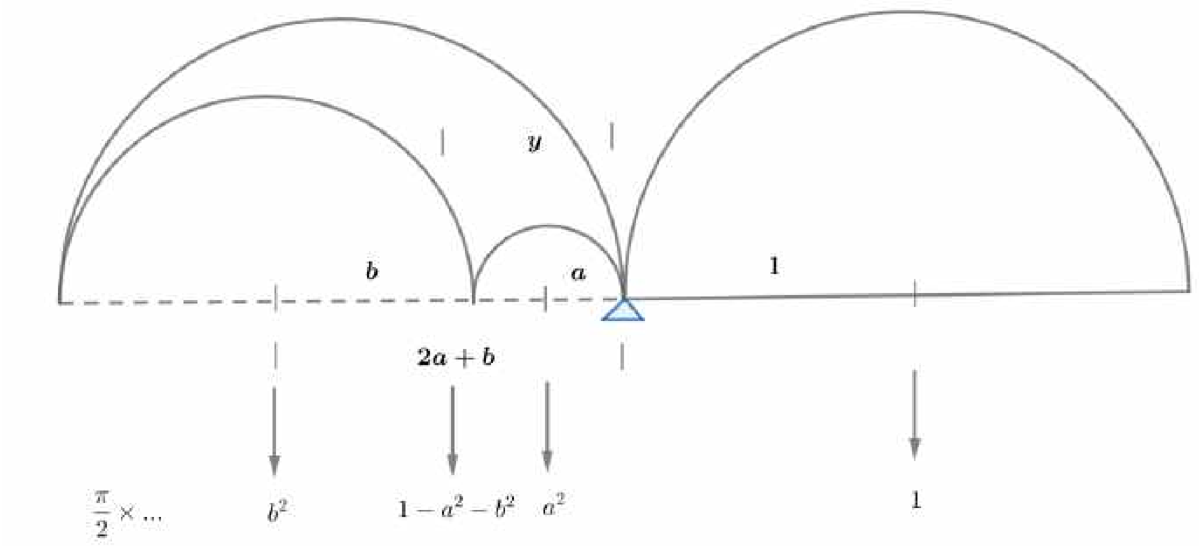
$$t(a+b)\frac{\pi}{2}(a+b)^2 = ta\frac{\pi}{2}a^2 + tb\frac{\pi}{2}b^2 + x\frac{\pi}{2}[(a+b)^2-a^2-b^2].$$

On simplification and substitution for *t*, we find $x=\frac{2(a+b)}{\pi}$, $=\frac{2}{\pi}$ for the unit circle.

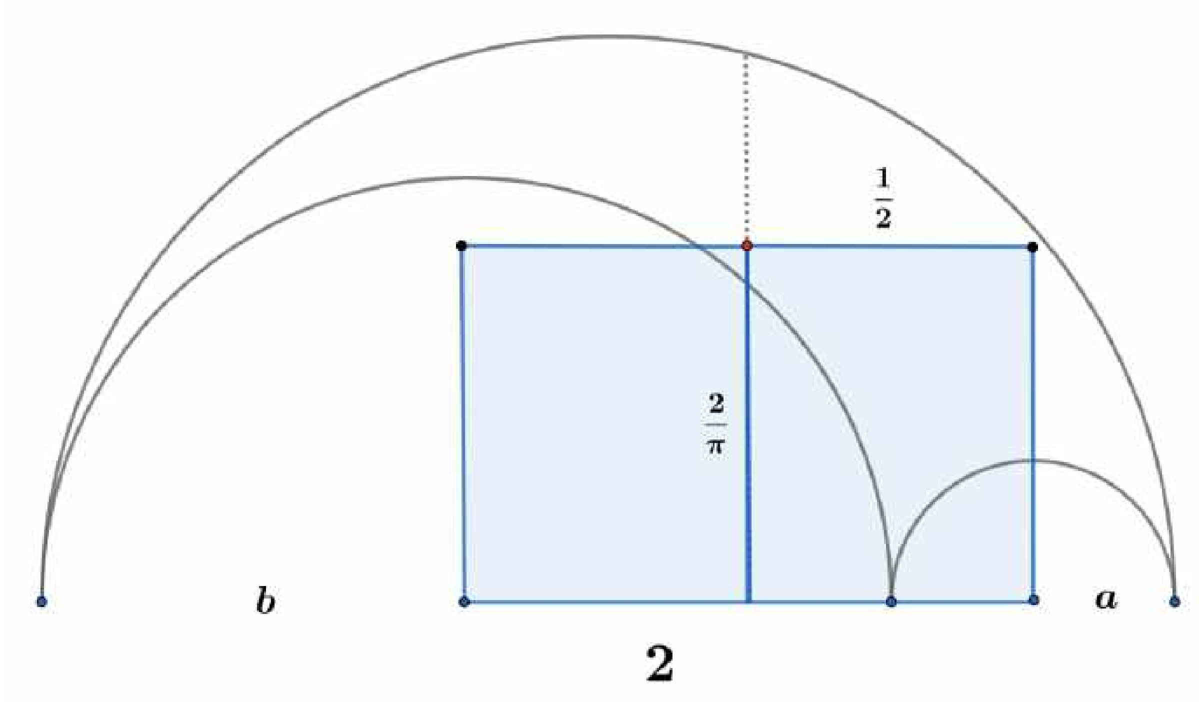
This is $\frac{3}{2}\times$ the distance from the diameter to the centroid of the big semicircle, a reasonable result.

x is independent of *a* or *b*. We can therefore take a **limiting case**, $a=1$ or $b=1$. The semicircle is now an arc, the half-circumference, and we have the centroid position for a thin wire bent in a half-circle.

In the *y*-direction we see from this balancing diagram that the position of the centroid depends on the values of *a* and *b*.



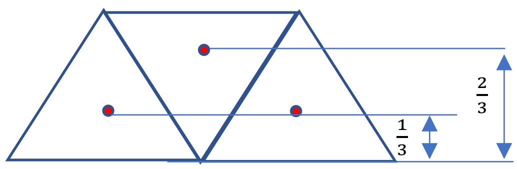
The solution of the resulting equation is $y = \frac{1-(2a+b)b^2-a^3}{1-a^2-b^2}$. If we turn the arbelos round, we have the other value, $y' = 2 - y$. The centroid lies nearer the smaller cut-out.
Tabulating results, we soon spot what we can confirm algebraically: $y - a = y' - b = 1/2$.
This result produces two little rectangles, each of area $\frac{1}{\pi}$:



Their combined area is therefore the reciprocal of the area of the big semicircle.

Returning to the centroid of the trapezium, we have a nice example of a **limiting case**.

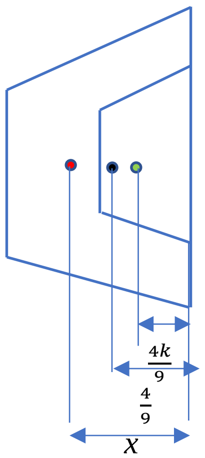
7) Our trapezium base is twice the top (as it is in half a regular hexagon). Applying the formula in **4**, we find the fractional height above the base is 4/9. But, since these three triangles are equal in area, all we need do is take the weighted mean of the centroid heights:



The fraction we want is

$$\frac{(2\times\frac{1}{3})+(1\times\frac{2}{3})}{(1+2)}=\frac{4}{9}.$$

Now we find the position of the centroid of this trapezium from which one of scale factor k has been removed. Below is our moments figure. Again we balance the components against the whole figure. Let the whole figure have unit mass and apply **similarity**.



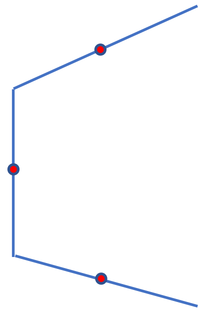
$$\frac{4k}{9}k^2+x(1-k^2)=\frac{4}{9}(1),$$

$$x=\frac{4(1-k^3)}{9(1-k^2)}$$

$$=\frac{4(1+k+k^2)}{9(1+k)}.$$

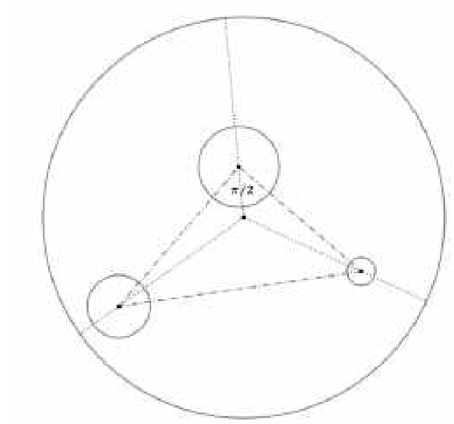
When $k=1, x=\frac{2}{3}$.

What we have here is a wire bent into three segments, whose centroids lie at their midpoints.



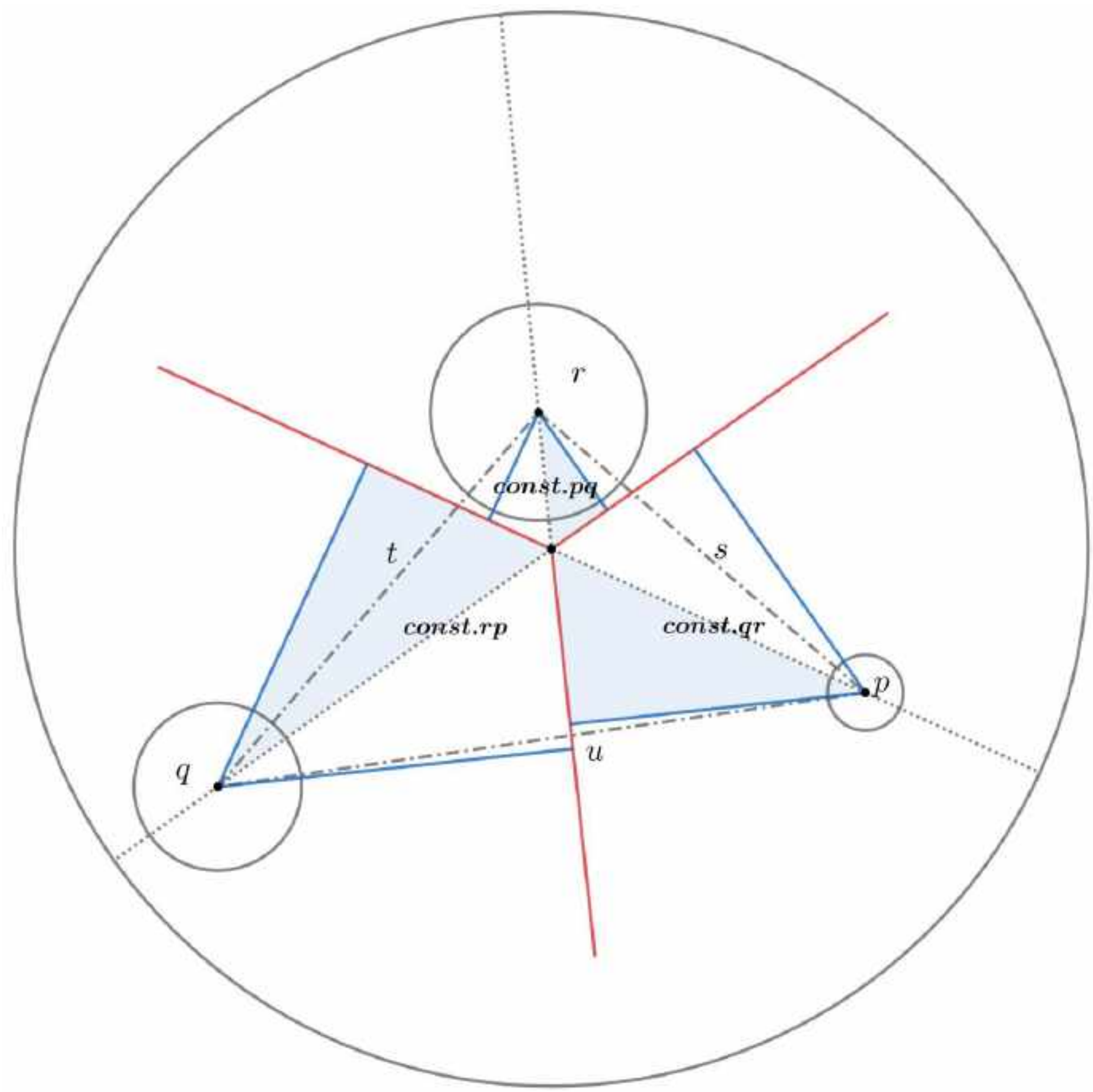
The weighted mean we require is $\frac{(2\times\frac{1}{2})+(1\times1)}{2+1}=\frac{2}{3}$, as expected.

7) Centred at points on three radii mutually inclined at $2\pi/3$, three circular holes are drilled through a uniform disk. The three masses removed have masses in integer ratio. The centres of the holes are vertices of a right triangle. The centroid of the perforated disk remains at the centre. Find the smallest ratio which meets these conditions and the dimensions of the corresponding right triangle.



Since the centroid does not move, it must be the same for the three masses as for the perforated disk. We can thus dispense with the disk and consider only the three masses.

We balance them in pairs about the radii which pass through the three centres (marked in red). The masses are *p*, *q*, *r*, the sides of the right triangle *s*, *t*, *u*. The blue lines mark perpendicular distances from the pivot lines. We see three similar kites each comprising a congruent pair of '1-2-$\sqrt{3}$' triangles. The length of the symmetry axis of one of these kites is the distance from a mass centre to the centroid. We can satisfy the moments condition by setting the *p* mass at a distance some constant × *qr* from the centre, and thus symmetrically for the other two.



By the cosine rule this gives:

$s^2 = (pq)^2 + (qr)^2 + pq^2r,$
$t^2 = (rp)^2 + (pq)^2 + p^2qr,$

$u^2 = (rp)^2 + (qr)^2 + pqr^2..$

Setting the third equation equal to the sum of the first two and simplifying, we are left with a quadratic equation. Taking the positive root, we have:

$r = \frac{p+q+\sqrt{p^2+10pq+q^2}}{2}.$

The expression is **symmetrical** in p and q. This is to be expected since we have not privileged either one in assigning labels.

By inspection, the smallest solution is $p = 1, q = 2, r = 4.$
(By **algebraic symmetry** and **dimensional consistency** there is an infinite number of scalings of this - (2,4,8), (3,6,12), (4,8,16), ..., which give the same triangle. Indeed there is

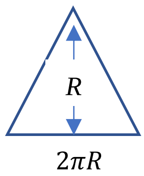
an infinite number of ‘primitive’ solutions, beginning (2,3,4), (5,6,15), (20,21,29), ... which can be scaled in the same way.)

Substituting back, we find the right triangle is the ‘1-2-$\sqrt{3}$’ case.

(J) Analogy

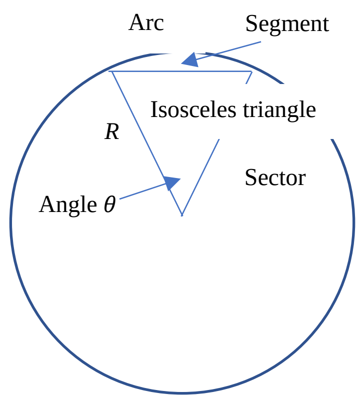
The drawing of analogies is central to mathematical thinking, indeed creative reasoning of any kind. Certainly any general formula implies an analogy between the special cases to which it applies. But it may be useful to explore how wide the range of cases may be. Analogies also hold across dimensions. And invariably the reasoning required in one dimension parallels the reasoning required in another.

1) A triangle with base *b*, height *h*, has area $\frac{bh}{2}$. If we divide a circle of radius *R* into infinitesimal annuli and lay them out straight, we have a triangle of base $2\pi R$, height *R*, and area $\frac{(2\pi R)R}{2} = \pi R^2$.

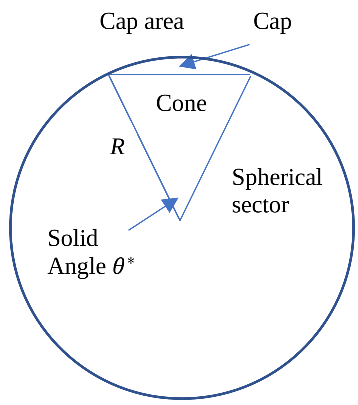


2) We move up a dimension. A cone with base radius *R* and height *h* has volume $\frac{\pi R^2 h}{3}$. If we divide a sphere of radius *R* into infinitesimally thick concentric shells and lay them out flat, we have a cone of base radius $2R$, height *R*, and volume $\frac{\pi(2R)^2 R}{3} = \frac{4\pi R^3}{3}$.
Note that this is the surface area of the sphere $\times \frac{R}{3}$ and that this multiplying factor applies to any spherical sector. This is the analogy with the 2-dimensional case:

Two Dimensions



Three dimensions



Arc length $= \theta R$

Sector area $= \frac{\theta R^2}{2}$

Sector area = Arc length $\times \frac{R}{2}$

Segment area = Sector area - Triangle area

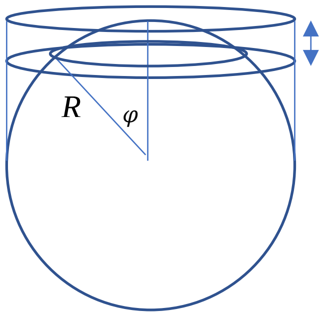
Cap area $= \theta^* R^2$

Sector volume $= \frac{\theta^* R^3}{3}$

Sector volume = Cap area $\times \frac{R}{3}$

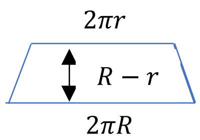
Cap volume = Sector volume - Cone volume

Rather than work with solid angles, we can use planar angles by virtue of Archimedes' 'hatbox' theorem:



Cap area = $(1-\cos\varphi)R \times 2\pi R = 2(1-\cos\varphi)\pi R^2$

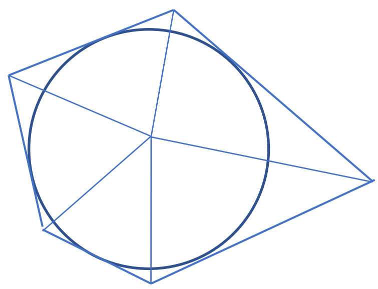
3) Correspondingly, the area of an annulus is that of a trapezium with these dimensions:



4) We move up a dimension. The volume of the frustum of a cone with base radius R, top radius r, height h, is $\frac{\pi(R^2+Rr+r^2)h}{3}$. If we divide a spherical shell of inner radius r, outer radius R, into infinitesimally thick concentric shells and lay them out flat, we have a cone of base radius $2R$, top radius $2r$, height $R-r$, and volume

$\frac{\pi[(2R)^2+(2R)(2r)+(2r)^2](R-r)}{3} = \frac{4\pi(R^3-r^3)}{3}$.

5) We can dissect a tangential polygon into triangles with apices at the incentre. Applying our area formula $\frac{bh}{2}$, we find that such a polygon with inradius r, perimeter p, has area $\frac{pr}{2}$.



We move up a dimension. The volume formula for a pyramid of base area a, height h, is $\frac{ah}{3}$. By analogy with the two-dimensional case, the volume of a tangential polyhedron with inradius r, total face area A, is $\frac{Ar}{3}$. The limiting case of such a solid is the sphere, whose volume must therefore be $\frac{(4\pi r^2)r}{3} = \frac{4\pi r^3}{3}$.

6) Moving in the opposite direction, for those familiar with calculus, we can say that, as the circumference of a circle is the derivative of its area with respect to its radius, $\frac{d(\pi r^2)}{dr} = 2\pi r$, the surface area of a sphere is the derivative of its volume with respect to its radius, $\frac{d\left(\frac{4\pi r^3}{3}\right)}{dr} = 4\pi r^2$.

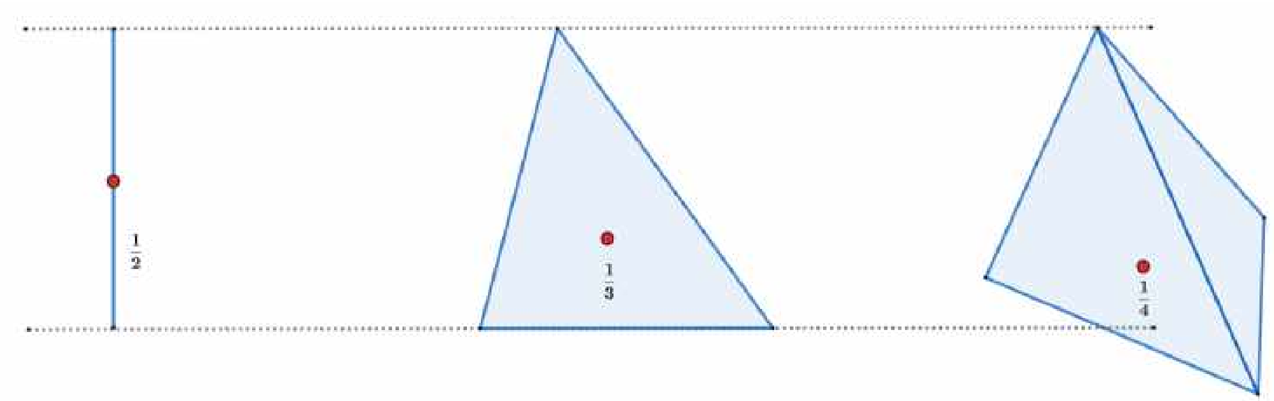
7) A simple numerical relation in one dimension invariably accompanies a simple numerical relation in another. The following dimensional sequence leads us to the height of the tetrahedron centroid above the base, (and by **Cavalieri's principle** to that of the cone, required above).

The fractional height of the centroid above the base is given in the following three cases.

A We have a straight bar of vanishingly small width.

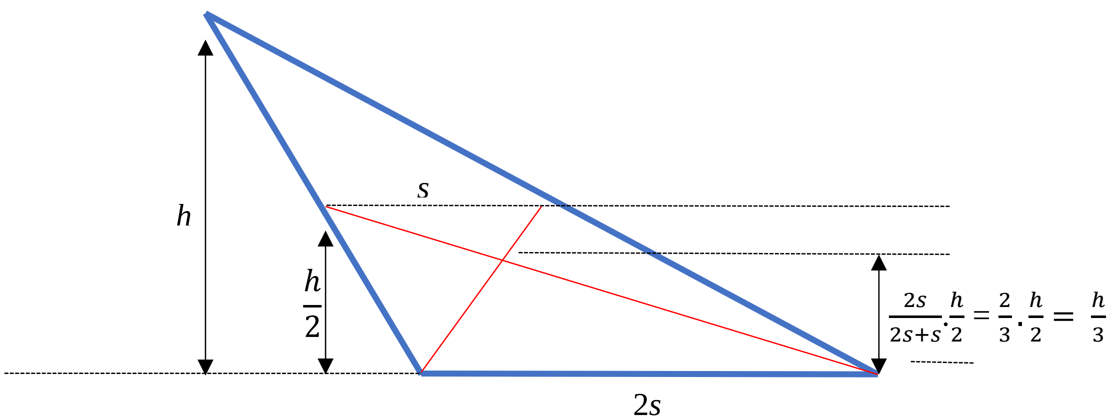
B We have a triangular lamina of negligible thickness.

C We have a tetrahedron.

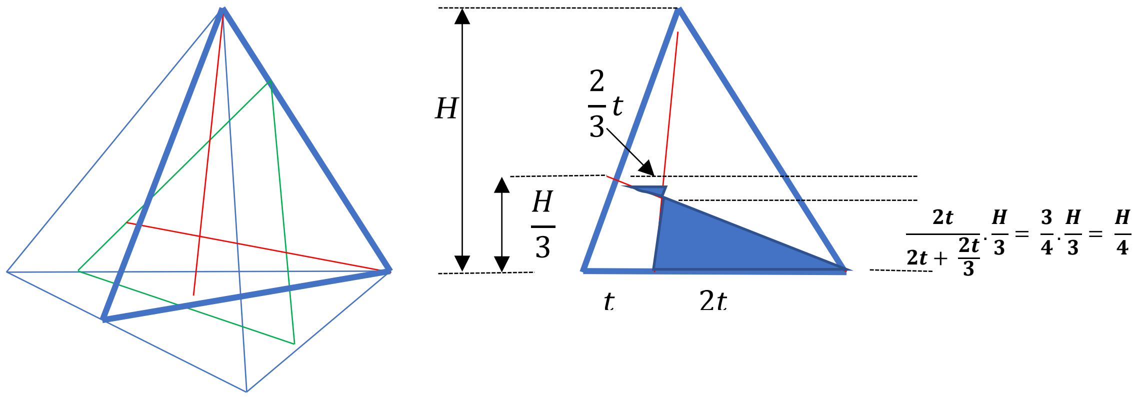


We shall assume result **A** to prove in turn results **B** and **C**.

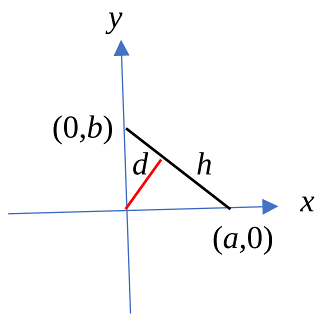
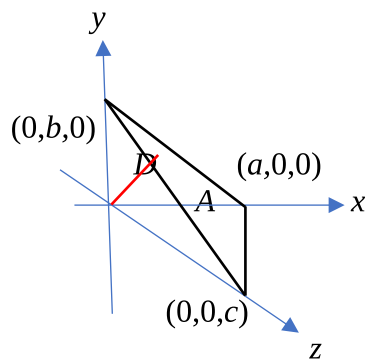
B We begin with a triangular sheet and take strips parallel to an edge. We know the centroid is at the centre of each, and therefore the centroid of the whole triangular sheet lies on a median. There are 3 such medians so the centroid of the sheet lies at the intersection of any two.
By similar triangles we see that the height above the base is 1/3 of the total height (and lies indeed 1/3 the way up any median).



C We slice our tetrahedron into a series of triangular laminae parallel to a face. We know that the centroid lies 1/3 the way up a median on each, so the centroid-of-centroids lies on a line joining the centroid of the top face to the opposite vertex. There are 4 such lines so the centroid-of-centroids lies at the intersection of any 2. By similar triangles we see that the height above the base is ¼ of the total height:



8) If we draw an analogy across dimensions, each object, and each attribute of that object, in one dimension must be matched by an object, and an attribute of that object, in the other. The analogy between Pythagoras' theorem in two dimensions and de Gua's theorem in three is clearest if we derive both using coordinate geometry.

Pythagoras	de Gua
Object: a right **triangle**	*Object*: a right **tetrahedron**
	
Intercept form of equation of ***line***:	*Intercept form of equation of* ***plane***:
$\frac{x}{a} + \frac{y}{b} - 1 = 0$.	$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$.
Distance d from origin:	*Distance D from origin*:
$\frac{1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2}} = \frac{ab}{\sqrt{a^2+b^2}}$. **(1)**	$\frac{1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}} = \frac{abc}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$. **(1)**
Equating ***areas***:	*Equating* **volumes**:
$\frac{hd}{2} = \frac{ab}{2} \Leftrightarrow h = \frac{ab}{d}$. **(2)**	$\frac{AD}{3} = \frac{abc}{6} \Leftrightarrow A = \frac{abc}{2D}$. **(2)**

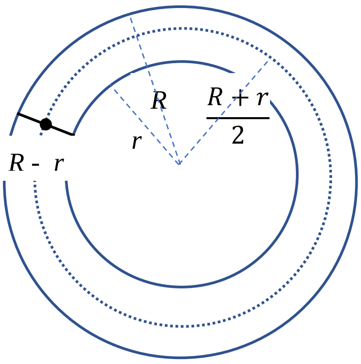
Substituting **(1)** *in* **(2)** *and squaring*:

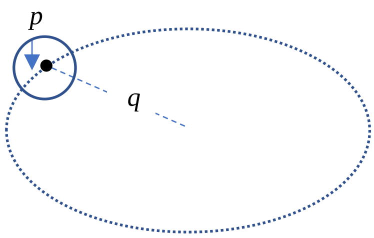
$h^2 = a^2 + b^2$.	$A^2 = \left(\frac{ab}{2}\right)^2 + \left(\frac{bc}{2}\right)^2 + \left(\frac{ca}{2}\right)^2$.

Lengths *of* ***sides*** *correspond to* ***areas*** *of* ***faces***.

Note that the dimension of length in Pythagoras' result is 2, but in de Gua's, 4.

9) Pappus' centroid theorems concern the area or volume of a shape generated by one rotation about an axis. In two dimensions, we obtain an area by multiplying the length of an element by the circumference of the circle followed by its centroid. In three dimensions, we obtain a volume by multiplying the area of an element by the same factor. We also obtain the surface area by multiplying the perimeter of the element. On the left we derive the area of an annulus, starting with a radially aligned line segment; on the right we obtain the volume of a torus, starting with a circle in the plane of the axis:

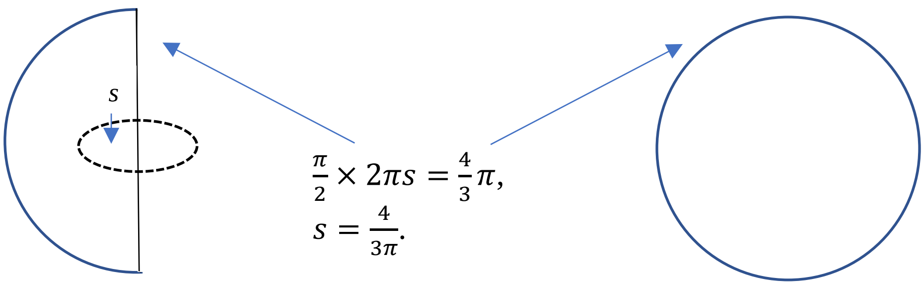




Area = $(R-r)\times 2\pi\frac{(R+r)}{2} = \pi(R^2-r^2)$

Volume = $\pi p^2 \times 2\pi q = 2\pi^2p^2q$
Area = $2\pi p \times 2\pi q = 4\pi^2 pq$

To find the distance between the centroid of a half-disk of unit radius and its diameter, *s*, we invoke the 3-D version. By rotating a semicircle about a diameter we generate a sphere:

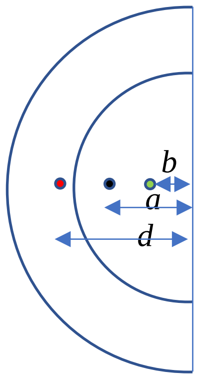


$$\frac{\pi}{2}\times 2\pi s = \frac{4}{3}\pi,$$
$$s = \frac{4}{3\pi}.$$

In the following example we invoke the principle of **moments** and the principle of **similarity** in an application of that result.

A half-annulus of unit outer radius is formed by extracting from a semicircle a concentric semicircle of half the area. Determine the distance *d* of the centroid from the diameter. (Compare **I.5**.)

We restore the semicircle removed and balance it against the half-annulus about the centroid of the complete semicircle (black). Since those regions are of equal area, the black-red and black-green distances are equal (*d*). (The figure does not represent the true relative distances, which have been increased for clarity.)



The radius of the semicircle removed is $\frac{\sqrt{2}}{2}$.

The distance of the black point from the diameter, a, is $\frac{4}{3\pi}$.

The distance of the green point from the diameter, b, is proportional to the radius of the semicircle removed,

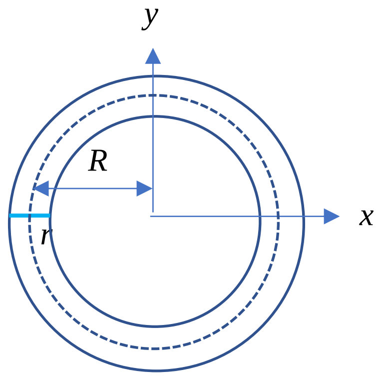
therefore $\frac{4}{3\pi} \times \frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{3\pi}$.

This makes $d = 2a - b = \frac{8-2\sqrt{2}}{3\pi}$.

Recalling our aside at **I.5**, when we calculate the fraction of the whole semicircle for which the centroid lies on the inner curve, we find that again, it's the 'small' golden ratio, τ. The circle removed at **I.5** and the semicircle removed here have the same radius.

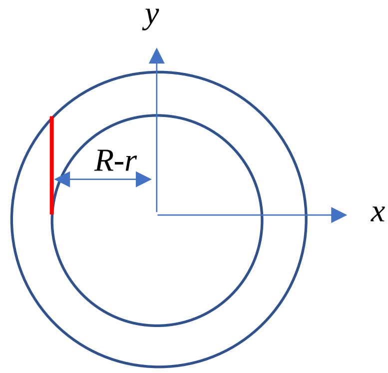
10) Returning to the torus, consider two different sections parallel to the axis of rotation symmetry, the *z*-axis,

one in the *x*-*z* plane:



This section is a circle.

... the other parallel to the *y*-*z* plane, tangential to the inner circle in this projection:

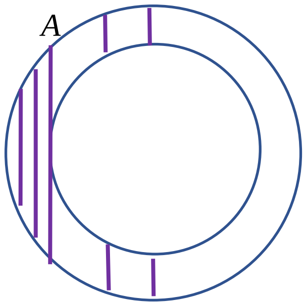


This is half a 'Cassinian oval' (a teardrop shape in this case).

From the Pappus theorems, we know that both the areas of the two section figures and their perimeters stand in the inverse ratio of their distances from the centre:

$$R - r \qquad : \qquad R$$

... this without knowing anything about Cassinian ovals.

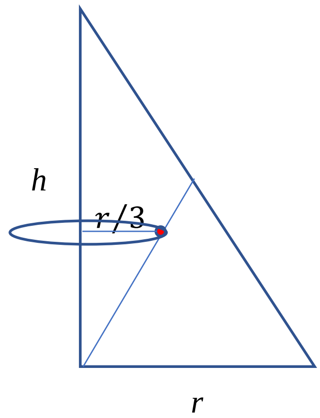


By the same token we know that, of the set of five regions to the left, A (a double teardrop) has the greatest area.

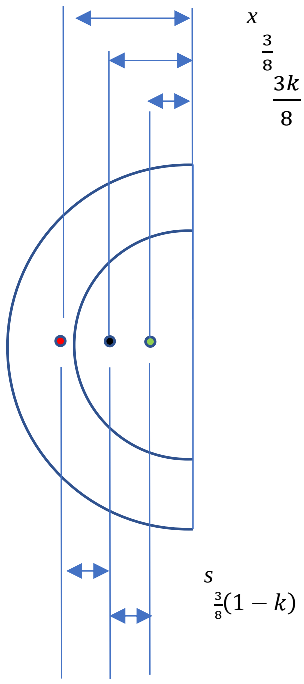
11) We can use a Pappus theorem to establish one of our standard formulas.

Using a result from **J.7**, and seeing by projection that
The radius of the circle to be followed by the centroid is
$\frac{r}{3}$, we have the formula for the volume of a cone:

$\frac{hr}{2} \times 2\pi \frac{r}{3} = \frac{\pi r^2 h}{3}$.



12) At **J.9** we find the centroid of a half-disk from which a smaller half-disk has been removed. In the three-dimensional **analogue** we have a hemisphere from which a smaller hemisphere has been removed. From **I.2** we know the position of the centroid in the complete hemisphere. Let the radius of the smaller hemisphere be *k* and the centroid of the shell be distant *x* from the diametric plane. Let $x - \frac{3}{8} = s$.



Dividing through by the volume of the hemisphere,
taking moments about the hemisphere centroid,
and applying similarity, we have:

$$s(1-k^3)s = \frac{3}{8}(1-k)k^3$$

$$\downarrow$$

$$s$$

$$\downarrow$$

$$x = \frac{3(1+k+k^2+k^3)}{8(1+k+k^2)}.$$

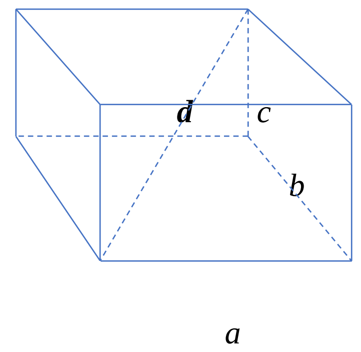
We check the limiting case $k = 0$ gives our original value.
At the other end of the scale, $k = 1$ gives $x = \frac{1}{2}$.
This represents a hemispherical shell of infinitesimal thickness.

This again is what we expect: by Archimedes' 'hatbox' theorem, projecting a hemisphere from its axis onto an enclosing cylinder preserves area, and we know that the centroid of a cylinder lies halfway along its axis.

Returning to **8**, here are two **analogous** problems in the 2-D and 3-D cases. The latter is more difficult only in that we must first ensure the constituent right triangles have integral area.

13)

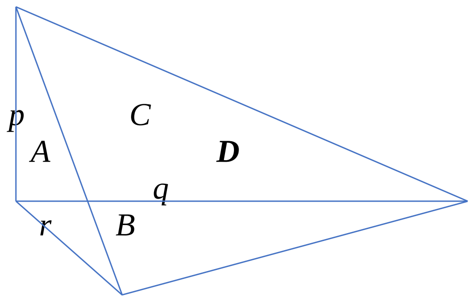
(i) The figure below is a cuboid. How can I choose integral lengths *a*, *b*, *c* so that *d* is integral?



The problem is that of finding a Pythagorean quadruple (*a*,*b*,*c*,*d*).

One family of solutions is
$[x, x+1, x(x+1), x(x+1)+1]$.

(ii) The figure below is a right tetrahedron. How can I choose integral areas *A*, *B*, *C* (labelling the right-angled faces) so that *D* is integral?



By de Gua's theorem the required quadruple is (*A*,*B*,*C*,*D*).
$A = \frac{rp}{2}, B = \frac{qr}{2}, C = \frac{pq}{2}$.
For the above solution family we need:

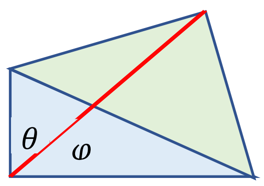
$2x = rp,$ **[1]**
$2(x+1) = pq,$ **[2]**
$2x(x+1) = qr\ .$ **[3]**
[3]$\div$ [**2**]: $x = \frac{r}{p}$. **[4]**

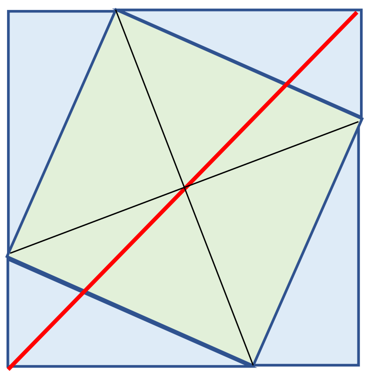
From **[1]**, **[4]**: $\frac{2r}{p} = rp, p = \sqrt{2}$.
So long as *q*,*r* also each have a factor $\sqrt{2}$ and *x* is an integer, our right triangles will have integral area, and we can obtain a solution set which belongs to the family above.

(K) The principle of maximum symmetry

Always draw a figure to make the symmetry clear, which may mean adding lines or siting the figure in a bigger one, possibly a tiling.

1) Here is a favourite proof without words of the late Ross Honsberger. Given these two right triangles, the green one isosceles, find angles θ and φ.

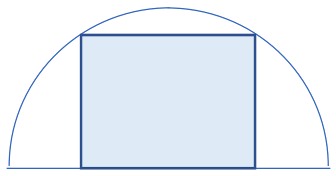


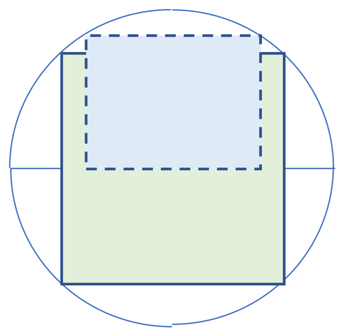


The principle is particularly useful in deciding on minimax cases.

2) A rectangle is inscribed in a semicircle. What is the aspect ratio which gives it the greatest area?

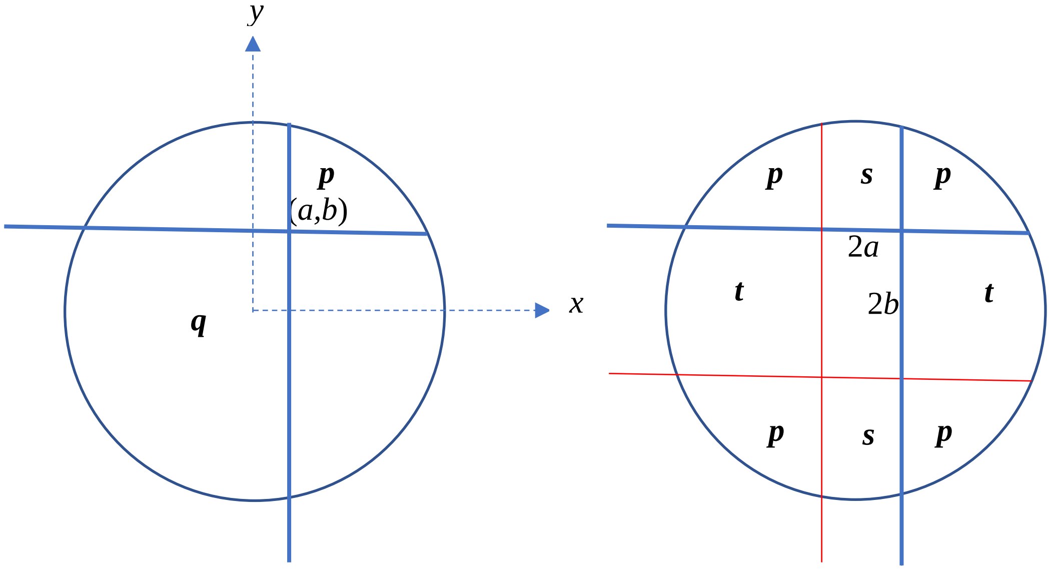
We know the answer for a full circle: 1:1, that of the inscribed square. If the answer to our first question differed from 2:1, it would contradict the answer to our second.





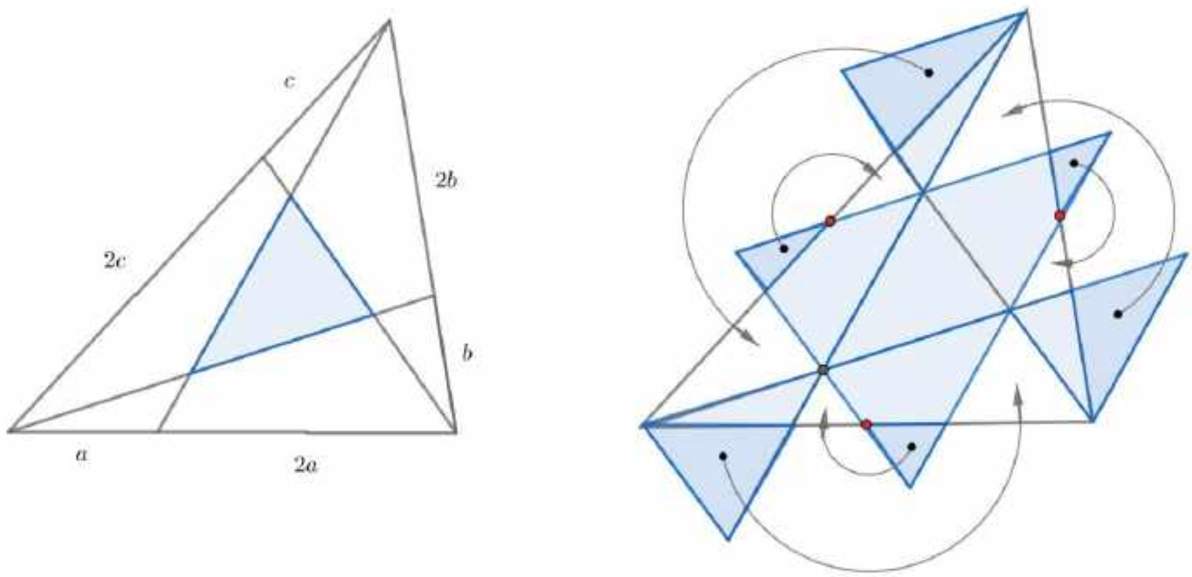
We can extend the argument. What is the greatest quadrilateral we can inscribe in a semicircle? Answer: Half a regular hexagon, where the cut is made along a greatest diagonal. What is the greatest pentagon in a semicircle? Answer: Half a regular octagon. And in general: What is the greatest n-gon? Answer: Half a regular $[2(n-1)]$-gon.

3) The circle on the left is cut by the blue cross so that the area q in the lower left quadrant is half the area of the whole circle. Write the area p in the upper right quadrant in terms of the coordinates (a,b) of the centre of the cross.

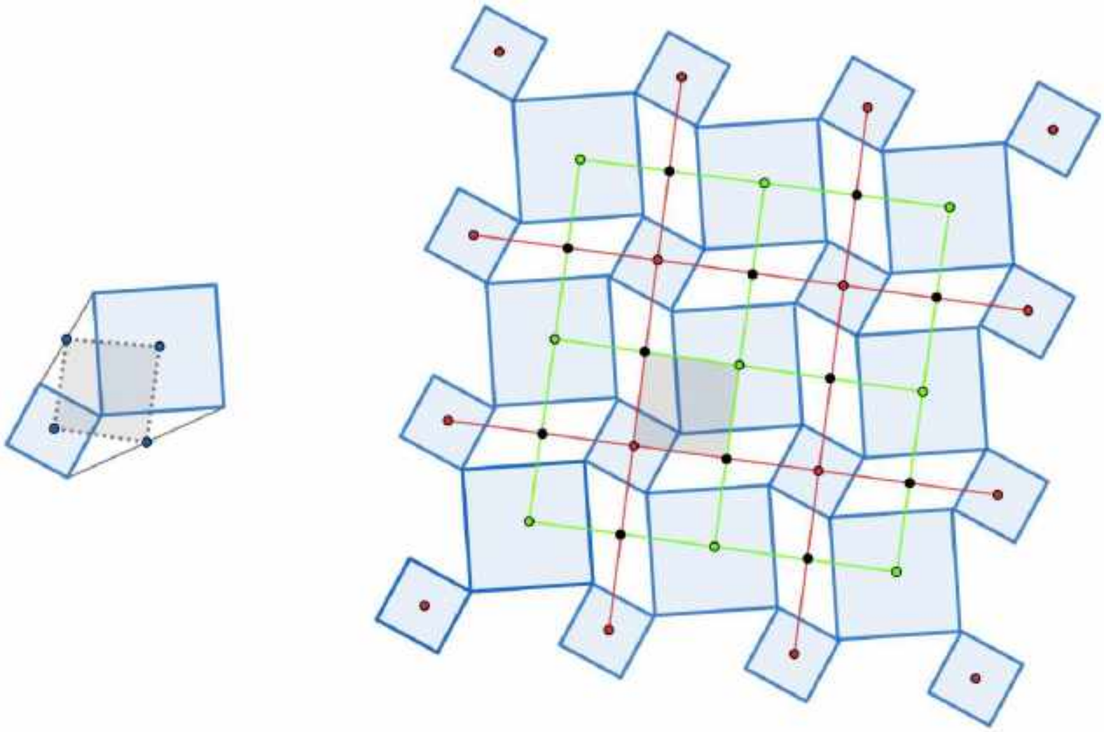


On the right we add the red lines to produce 9 regions symmetrical about the centre. Equating areas, we then have $q = p + s + t + 4ab = 3p + s + t \Leftrightarrow p = 2ab$. (Bur see **M.**)

4) As a special case of Routh's theorem, we find that the central triangle on the left has 1/7 the area of the outer one. J. G. Mikusinski drew two parallels to each side of the inner triangle, which he extended. This created 7 congruent triangles, the central one and 6 more, parts of which lay outside the big triangle. But, with the half-turns shown, he swung the parts outside over the gaps inside, filling the big triangle and confirming the result.



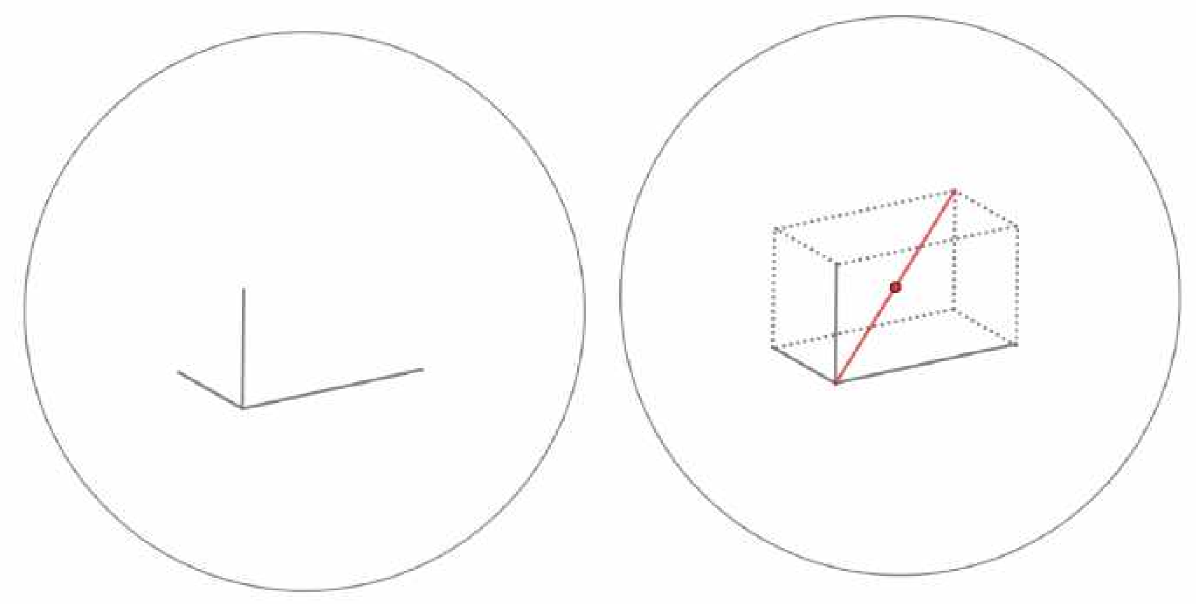
5) Case **4** is an example of embedding a figure in a tiling. Though no measures are involved, here is another. In the figure on the left are marked two square centres and two midpoints of lines joining corresponding vertices. The Finsler-Hadwiger theorem claims that the dotted quadrilateral is a square.



We have produced the tiling on the right by translating that figure. The line midpoints now appear as centres of parallelograms. The red and green lines are parallel. The red and green points are centres of fourfold rotation symmetry. Therefore both red and green quadrilaterals are squares and, by virtue of the translation, of equal size. The black points are centres of twofold rotation symmetry. Because they lie at midpoints of red and green sides, they define squares like that shaded which have half the edge of the larger squares. Our original quadrilateral is such a figure.

6) We are asked to find the position of the centre, and the radius, of the sphere circumscribing a right tetrahedron of perpendicular edge lengths *a, b, c*.

Four points, not all in one plane, define a sphere. The tetrahedron, and the cuboid defined by it, must therefore share the same circumsphere. This locates the centre and gives us a radius equal to half the length of the cuboid's space diagonal.



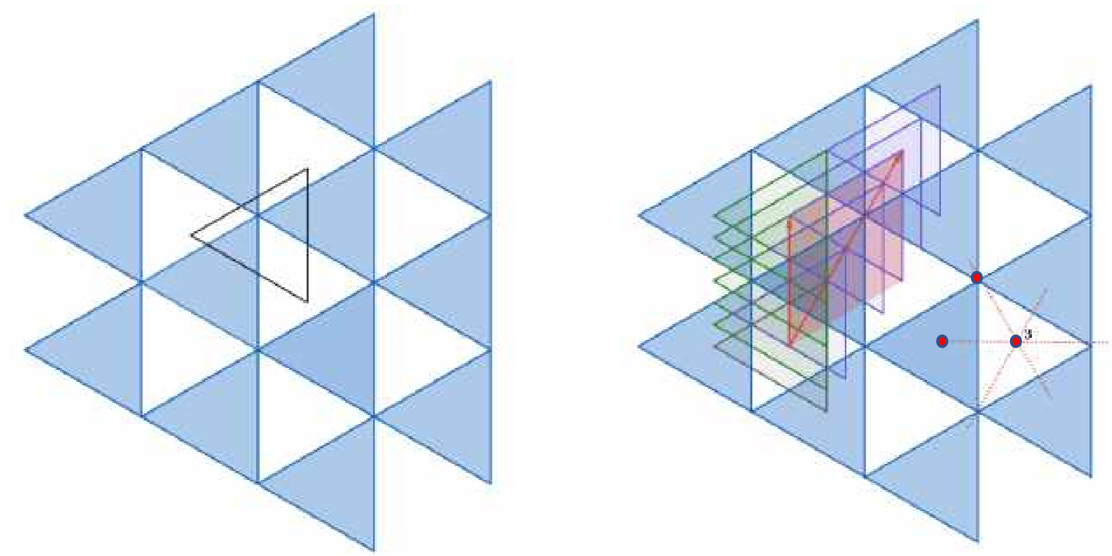
Analogy with the right triangle, where the circumcentre is the midpoint of the hypotenuse, would lead us to expect that result.

Establishing the global symmetry can help us restrict the solution space.

7) This isometric grid is of two colours. Triangles of opposite colour share an edge. We can move the free triangle shown only by translation. What is the least fraction, *f*, of the triangle which can be blue?

We shall set up a hypothesis then test it.

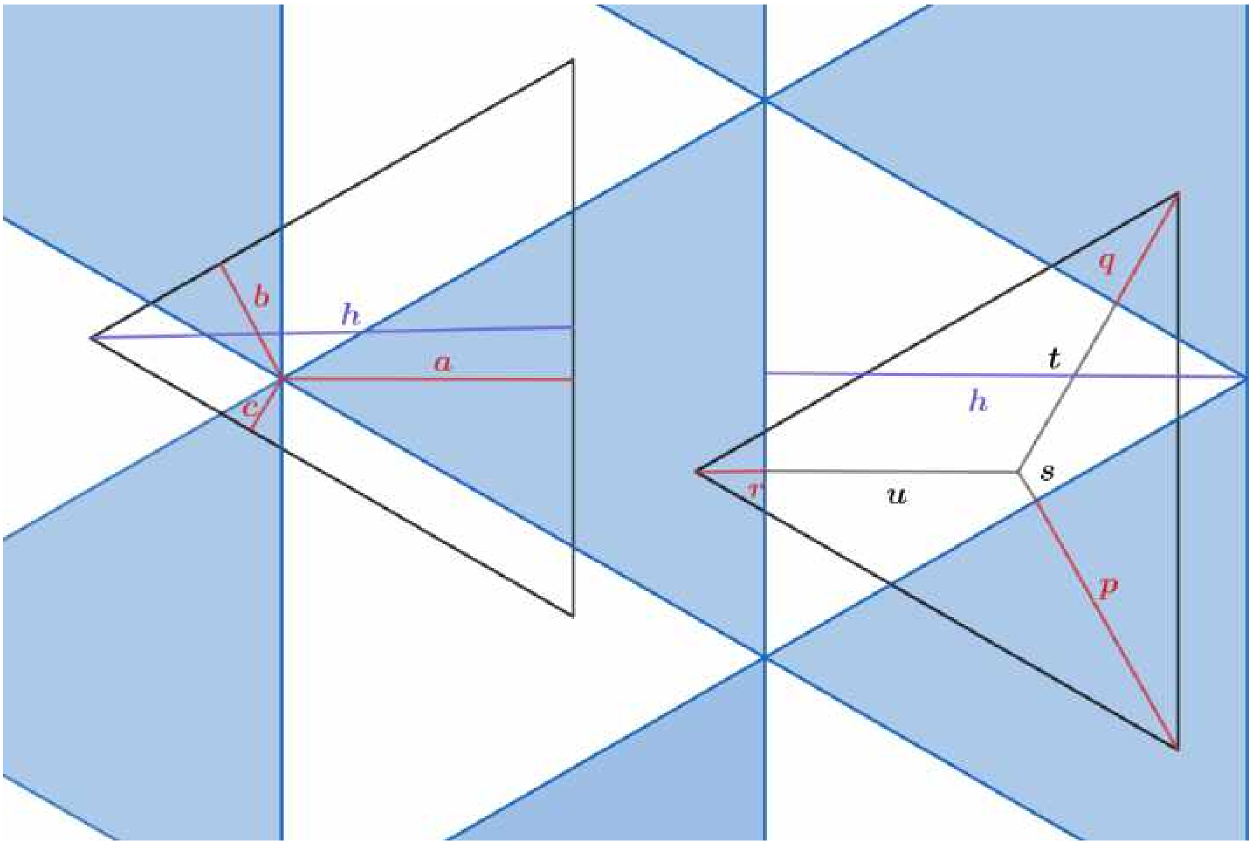
A preliminary observation is that white triangles point right, the free triangle points left. Since the free triangle cannot therefore be brought into coincidence with a white triangle, *f* must be greater than zero.



On the right we show in red a unit translation cell of the tiling. We also show the locations of the three centres of 3-fold rotation symmetry and the mirror lines. As a result of the symmetry, if we move the centroid of the free triangle along the vertical arrow, that is, in line with a triangle side, and along the inclined arrow, that is, at right angles to a triangle side, we have information about 12 directions mutually inclined at 30°. The 6 distinct positions we've chosen in the two directions enable us to work out *f* in $1/36\,s$ by counting triangles on an isometric grid where a tiling triangle side is 6 times the length of a grid triangle side. The sequence of values along a vertical line runs: $1/3, 7/18, 5/9\,, 1/2\,, 5/9\,, 7/18$; along an inclined line: $1/3, 1/2\,, 1/3, 1/2, 1, 1/2$. These figures suggest the hypothesis that the answer we seek is 1/3. We now show that.

We can imagine a continuous, centrally symmetrical surface in which each of these fractions are heights above the base plane. The simplest possibility is that the heights in a section between any two of our discrete points fall between the start and end values. That would establish our claim that we have found the global minimum. We shall not try to do this but instead examine two local minima, using ideas from sections **(B)**, **Algebraic symmetry** and **(G)**, **Similarity**, which, between them, cover all cases.

We meet the fraction 1/3 in two situations: in one, the centroid of the free triangle coincides with a tiling vertex; in the other, it coincides with the centroid of a white tile. We shall show that, when the free triangle contains a tiling vertex (below left), the smallest fraction,f_1, is 1/3; and that when its centroid falls within a white tile (below right), the smallest fraction, f_2, is also 1/3.



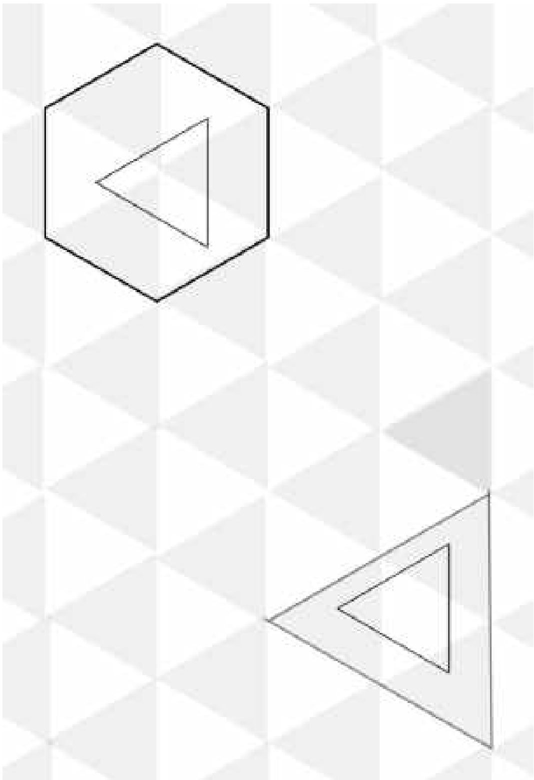
If we give a triangle unit area, we have by similarity:

$f_1 = \frac{a^2+b^2+c^2}{h^2}$. This expression is **symmetrical** in *a*, *b*, *c*. Therefore any value of a which gives a minimum value for f_1 must equal that of b and *c*, call it *m*.
By Viviani's theorem we have $a + b + c = 3m = h$. Thus $f_1 = \frac{3m^2}{(3m)^2} = \frac{1}{3}$. (In fact we don't need to invoke the theorem, we only need observe that, when $a = b = c$, each $= \frac{h}{3}$.)
$f_2 = \frac{p^2+q^2+r^2}{h^2}$.

$s + p = t + q = u + r = \frac{2h}{3}$
$\Leftrightarrow (s + t + u) + (p + q + r) = 2h$.
By Viviani’s theorem (or otherwise) $s + t + u = h$.
Subtracting, we find $p + q + r = h$ and we can argue as above.

In the figure below we see that, in the first case, the locus of the boundary of the free triangle is a hexagon with a side equal to a grid triangle side.

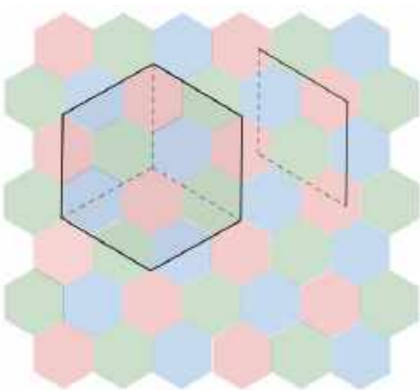
In the second case we have extended the locus so that the boundary is that of a triangle with twice the edge of a grid triangle. It represents all the positions where the free triangle does not contain a grid vertex. We note that, when the centroid of the free triangle moves into a grey zone, the amount of grey it contains becomes at least 1/2, that is, greater than 1/3.



Since the hexagon allows all positions of the free triangle where it contains a grid vertex, and the big triangle allows all positions of the free triangle where it does *not* contain a grid vertex, between them, the hexagon and the big triangle allow all possible positions of the free triangle. Since in each case the grey fraction is not less than a third, this local minimum must be the global minimum.

A simpler case is this.

8) What can you say about the fraction of this hexagon occupied by a given colour as it is translated over the tiling?



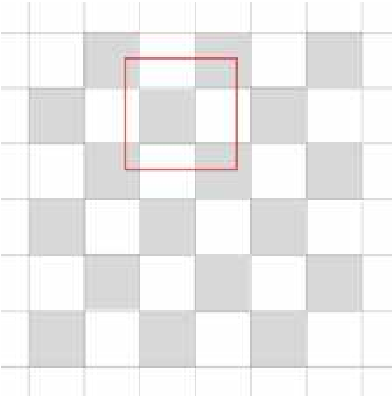
Dissecting the hexagon into three congruent rhombuses, we note that each is a unit translation cell for the whole tiling.

In any given position therefore it must contain the same proportions of the three colours.

Since the three colours are equally represented, the fractions of the cell must be equal, therefore $\frac{1}{3}$.

And this must be true of the hexagon the three rhombuses compose.

9) Show that, wherever we translate this square on the checkerboard, exactly half its area is black, half white.



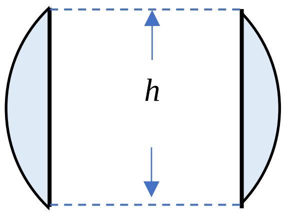
We can make the same argument we did in **8**. The square comprises two unit translation cells.

We could also make the case for the large hexagon in **7**.

(L) The complete information principle

In the real world the problems we face may be poorly defined and the information needed to solve them incomplete. But when we are set a question in an exam or competition, we must assume that enough information has been provided to answer it. On occasion, a fact may be implied but not stated: we can 'read between the lines'. A famous instance is the following.

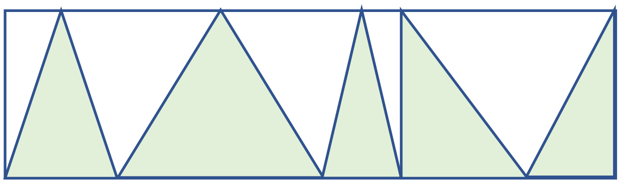
1) A metal napkin ring of height h is made by symmetrically boring out a cylinder from a sphere. (The result is shown in axial section below.) What is the volume of metal remaining?



We are not told the radius of the cylinder, which must therefore cancel out in any calculation we may make. We are therefore free to make it zero. This leaves us with a sphere of diameter h, volume $\frac{\pi h^3}{6}$. (This result is justified at **D.5**.)

The following example is due to Andrew Jeffrey.

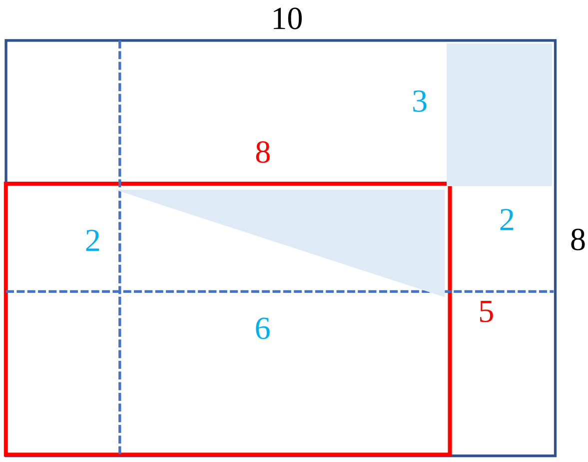
2) This rectangle has area A. What is the area shaded?



We see that the triangles all have the height of the rectangle but different base lengths, which, however, sum to the rectangle's width. We can therefore combine the triangles in a single one whose base has the width of the rectangle, giving our answer, $\frac{A}{2}$.

(M) The principle of assuming the minimum

We invoke this principle most commonly when reviewing a result already obtained. Looking over **K.3**, we realise that all we have assumed about the closed curve is the two perpendicular mirror lines. The result therefore applies equally to this figure:

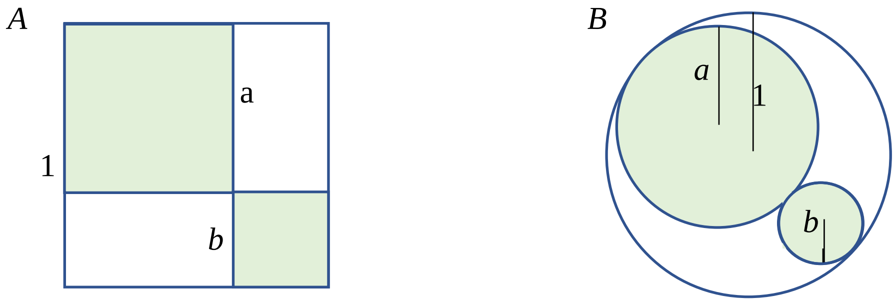


The red rectangle has half the area of the outer rectangle. The four corner rectangles are congruent. Any one of them is equal in area to the right triangle.

Recalling **L**, **The complete information principle**, we realise that in **K.3** we have *too much* information, that is to say, we have a special case of a more general one. The statement of the problem would be improved by adding something like ‘Can you extend your result?’. This criticism applies also to **H.3**, where the rectangle stands for any parallelogram, and **G.3**, where the cube stands for any parallelepiped.

Summary

We end with two examples which apply a number of the principles we have been discussing. Throughout these notes our aim has been to minimise computation. The principles allow us to steer a course between algebra and geometry which achieves this. In both examples we are interested in the fraction *f* of the outer figure occupied by the two inner ones.



In each case we have three similar shapes, the two within sharing respectively the diagonal and a diameter of the third. By the principle of **similarity** $f_A = f_B = f = \frac{a^2+b^2}{1^2} = a^2 + b^2$. Notice that we have observed **dimensional consistency**. Though a and b alone have the dimension of length, *f* is a ratio, therefore dimensionless. By the **relabelling** principle we can swap a and b without changing the figure: the expression for *f* must therefore display **algebraic symmetry**, and we see that it is indeed symmetrical in a and b. Consider *f* as the function $g(a, b)$. Since $a = 1 - b, b = 1 - a$, there is a symmetrical relation between a and b. We can choose either to write $g_a(a) = 2a(a - 1) + 1$, or $g_b(b) = 2b(b - 1) + 1$. To find the maximum value of the function, we take the **limiting case** $a = 1, b = 0$, or $b = 1, a = 0$, giving the value 1. By the **relabelling** principle, were the minimum value other than ½, swapping a and b would produce two minima. Assuming there is a unique minimum, it must be ½. Notice that this argument avoids any analysis of the function $g(a, b)$.

Occurrences of a given shape:

Shape	Type	Attribute	Text location												
			A	B	C	D	E	F	G	H	I	J	K	L	M
Triangle	General	Area		3			1			5		1			
Rectangle										5					
Parallelogram										5					
		Side length relations													
		Centroid position										7			
		Area of triangle defined by cevians											4		
Trapezium	General	Area								5		3			
		Centroid position									4				
Cone	General	Volume				1	3			1,7	1	1,11			
	Circular	Surface area					5								
	General	Centroid position				3			6			5			
Cone frustum	General	Volume	1	1		2			2			4			
	Circular	Surface area					3								
	General	Centroid position			2						3				
Cylinder		Volume				5					1				
		Surface area					3								
		Centroid position									2				
Cylinder pair	Axes cutting at right angles	Volume of common solid				4									
Prism	Triangular									1					
Pyramid	General	Volume		2						1		5			
		Centroid position													
Pyramid frustum	Square	Volume				2				2					
		Centroid position													
Cone	Circular	Surface area				5						2			
Conical shell											5				
Circle		Area					3			5	1	1,3,9			
		Circumference										6			
Annulus		Area					3				1	3,9			
Half-annulus		Centroid position										9			
Semicircle		Centroid position													
Circular arc		Length										2			
Semicircular arc		Centroid									6				
Circular sector		Area										2			
Circular segment		Area										2			
Lune	Crescentic										5				
Ellipse		Area								5					
		Curvature radii			1										
		Focal property					4								
		The orthoptic circle						2							
Sphere		Volume				4,5				5	1	2,5,6			
Sphere less central cylinder		Volume												1	
		Surface area										6			
Spherical shell		Volume										4			
		Centroid position									5				
Spherical sector		Volume										2			
Spherical cap		Surface area										2			
		Volume										2			
Spherical segment		Volume								7					
Torus		Volume										9			
		Surface area													
Ellipsoid		Volume								5					
Hemisphere		Centroid position									2				
Hemispherical shell	Empty	Centroid position										12			
Tetrahedron	Regular	Volume								4					
Tetrahedron	General	Centroid position								1		7			
Triangle	Right	Area			3										
		Pythagorean relation										8			
		Circumcentre position													
		The lunes of Alhazen							4						
Tetrahedron	Right	Volume			3							8			
		Pythagorean relation										8			
		Circumcentre position											6		
		Circumradius											6		
Tangential polygon		Area										5			
Tangential polyhedron		Volume										5			
Cube		Volume								6					
Cubic shell		Volume								6					