SYMmetry

Newsletter of the Society of Young Mathematicians Autumn 1994

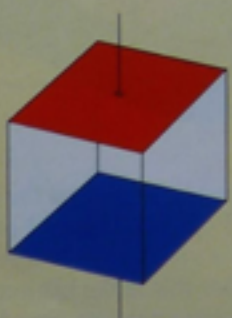
NOTE THIS DATE!

The next major SYMS event will be a Maths Fair to be held at the University of Sussex near Brighton on Saturday 8th April from 10.30 am to 4 pm. It is open to both members and non-members – school parties will be welcome.
Enjoy a day of mathematical fun for all the family!

COLOURS & MAPS

In a recent newsletter we were asked, "How many ways are there to colour a cube (tetrahedron, octahedron, . . .) using each colour once?" If you tackled the problem, you'll soon have realised that it isn't the (simpler) one of colouring a row of boxes:

Here, there are 6 choices for the first box, 5 for the second, 4 for the third, . . . so that the number of ways is equal to 6 x 5 x 4 x 3 x 2 x 1 ("factorial" 6, written "6!"). The symmetry of the shape means we have, as it were, to lock it in position before we start colouring. For example, if we colour opposite faces of the cube red and blue:



we can rotate it about the marked axis into 4 positions and it looks exactly the same. This means that, if we colour one of the remaining faces yellow, we might equally well have chosen any of the other 3 for this colour.

However, this freedom is not unrestricted. You may have found from your investigations that colouring a solid with n sides, using each colour once, is a multiple of $(n-3)!$ (Did the number you found for the tetrahedron divide by $(4-3)! = 1! = 1$? Yes, it did! Did the number you found for the cube divide by $(6-3)! = 3! = 6$? Did the number you found for the octahedron divide by $(8-3)! = 5! = 120$?)

We were also asked to think about ways of colouring a solid where every face was not necessarily a different colour. A particularly interesting way is to "map-colour" it: I'll explain what I mean by this.

You may have heard of the 4-colour map theorem. This says that no map, however complicated, needs more than 4 colours to ensure that countries touching are coloured differently. This doesn't just apply to a flat map:

We can bend it round a sphere – it doesn't matter that it gets crumpled – so that the country surrounding all the others (or it might be the ocean surrounding a continent) acquires a boundary:

Now, the "sphere" can be a polyhedron. And on this, each country can be a face. Our requirement becomes: no faces touching edge-to-edge may be the same colour. Because we know that we shall never need more than 4 colours we can build our solids from Polydron or Clixi since shapes in both kits are supplied in red, green, blue and yellow.

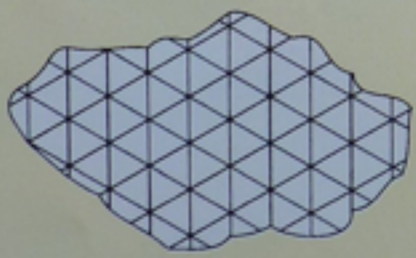
What about the regular solids: how many colours are needed for the tetrahedron? the cube? the octahedron? The colouring of the pentagonal dodecahedron is particularly beautiful: each colour is symmetrical about a rotation symmetry axis of order 3:

But there's a lot more to discover. How many different colour assortments meet at edges and how many times does that kind of edge repeat? How many meet at corners ("vertices") and how many times does that kind of vertex repeat?

What about solids where every vertex looks the same and every face is a regular polygon but they are of more than one kind ("semi-regular" solids)? What about solids made from rhombuses? What about . . .?

Once you've found the smallest number of colours needed for your shape, ask, "How many different ways are there to colour my shape in this way?" Also ask, "How do I know that this solid is going to need at least 3 colours while that one may only need 2?"

In the same newsletter we were looking at tessellations. This is one of the 3 "regular" ones and doesn't look very interesting . . .



but imagine that you map-colour a regular tetrahedron with wet paint and sit it on one of those triangles. You then roll it over edges so that it lands on adjacent triangles and colours them. What results? Build your tetrahedron in Polydron or Clixi. Roll it. Using the same kit, make the carpet it prints.

The tetrahedron is the only regular solid which prints a carpet without smudging. Prove that the others fail. Does/do any other solid/s print without smudging and cover the plane without gaps? If you think the answer is "Yes", try to find it/them; if you think the answer is "No", try to prove it.

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MAP-COLOURING POLYHEDRA
Part 1

Readers will have heard of the Four Colour Map Theorem, which states that no more than 4 colours are needed to ensure that no 2 countries sharing a border have the same colour. We usually think of a flat map but the theorem applies to any surface of the same 'genus', i.e. of the same kind from the point of view of topology – often described as 'rubber sheet geometry'. The sphere is such a surface. So are polyhedra.

In **Part 1** of this article we shall look at certain solids, asking, 'What is the smallest number of colours needed?' While doing so, we shall find a couple of useful properties. But then I shall issue some challenges. I hope you will meet these first with some mental geometry. If, like me, you then find you need to make models, the most convenient material is Polydron, in particular, Polydron Frameworks: this allows you to build skeleton solids and thus view all the colour relations without turning the models round.

Many books and posters show you the 5 regular solids and some of the semi-regular ones. My bible is:

'Mathematical Models', H.M.Cundy & A.P.Rollett, Tarquin.

My pages references will be to that book.

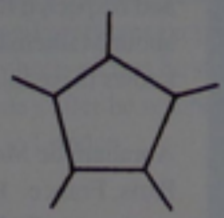
In mathematics we distinguish 'necessary' and 'sufficient' conditions. For it to rain, it is necessary that there are clouds in the sky. But that is not sufficient: a cloudy day may pass without a drop of rain falling. In a tetrahedron 3 faces meet in a vertex; the same is true of a cube. 3 is an odd number. We need 3 colours at such a vertex, otherwise the same 2 will meet when we complete a circuit:

This number is also sufficient to map-colour the cube, not so for the tetrahedron:

The case of the tetrahedron illustrates another property. Think of the tetrahedron as a corner cut off some solid – this process is called 'truncation'. Whereas 3 colours were needed before, the common face produced by truncation requires a 4th:

More generally, truncation increases the number of colours needed at any vertex by 1.

Look down on a face of the regular dodecahedron (p.87)**. From what we have discovered already, how many colours do you know it will need and why?**



Next challenge:

How many colours does the regular icosahedron (p.88) **need?**

If you make a model, save it till you've read **Part 2** because I want to show you something surprising about it.

We said that truncation increased the number of colours needed *at any vertex* by 1.

Is the same necessarily true for the solid as a whole?

Examine the icosahedron (p.88) and *truncated* icosahedron (football or 'Bucky ball') (p.110) on the one hand and the dodecahedron (p.87) and *truncated* dodecahedron (p.109) on the other. (There are – as yet – no Polydron decagons but pictures may be enough to tell you what you need to know. If you want to make a model, the most convenient polygons are the mats – *m*athematical *a*ctivity *t*iles – produced by the Association for Teachers of Mathematics. You stick these together with Copydex or other P.V.A.-based adhesive.)

In **Part 2** I shall describe how this article was inspired by the work of 2 year 9 students at The International School of the Hague – and perhaps include some of your own findings? E-mail me.

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Ed.

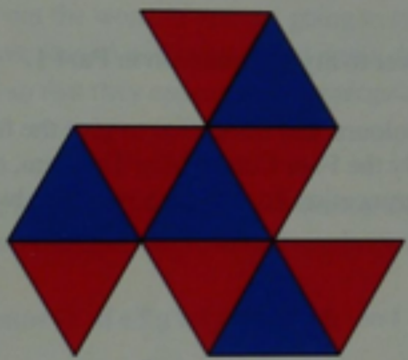
MAP-COLOURING POLYHEDRA

Part 2

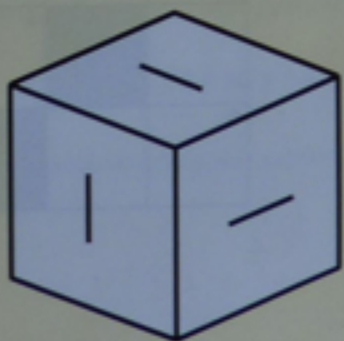
In my roadshow I challenge the visitors to map-colour 3 regular solids: the octahedron with 2 colours, the cube with 3 and the dodecahedron with ...?

4: it needs the maximum number.

At the end of **Part 1** I mentioned the work of 2 students. Having solved the dodecahedron puzzle, Giang came up with what he thought was an algorithm for deciding the 'chromatic' number – the number of colours needed – for any polyhedron, using only the number of faces. Meanwhile Marius decided to tackle a solid I had not suggested: the icosahedron. Knowing how the icosahedron was constructed but without a net in front of him, Marius grew his own outwards from 1 face. I show it when it had reached 10 faces – in other words, at the halfway point. Notice the 3-fold rotation symmetry, also to be found in the folded solid. He realised the net itself, being 'connected' – every face shared at least one edge with another, required 2 colours:



His plan was to fold the net, substituting a 3rd colour every time 2 faces of the same colour collided. To his disappointment he failed to close the whole net without introducing a 4th. My own memory was that 3 colours were possible. I sketched a solution and, after Marius had left that day, assembled it. To my delight, a feature appeared in the model which had not been apparent in my sketch. The dodecahedron has 12 faces. 12 divides nicely by 4 to give 3 faces of each colour. The icosahedron has 20 faces. 20 does not divide nicely by 3. We might expect the colours to be distributed 7,7,6 or 8,6,6. Which does symmetry prefer? Answer: 8,6,6. An interesting property of the icosahedron (shared with its 'dual', the dodecahedron) is that it fits snugly in a cube with edges aligned like this:



Say each of these edges joins a red triangle and a blue one. How many faces is that altogether?

Answer: 6 (cube faces) x 2 = 12.

That leaves 20 – 12 =8, one for each cube vertex.

The 6 bent red-blue rhombuses are separated by the 8 vertex triangles, so we only need a 3rd colour – green, say – for the whole solid.

I sent my model to Marius via his teacher. When he returned next day, he reassembled it from memory.

Now a problem arose. Giang's algorithm predicted that an icosahedron should need 4 colours. But here was a 'counterexample' – and I pointed out another model in the room which upset his reasoning: the rhombic dodecahedron, which, unlike its pentagonal brother, only needs 3 colours. I suggested it was not enough to consider only the number of faces a solid had, but how they were joined: the pentagonal dodecahedron has only 3- vertices; the rhombic dodecahedron, 8 3- and 6 4-.

If Giang modifies his algorithm to take that into account, can it successfully predict the chromatic number for *all* polyhedra?

We shall see: I've left him to consider.

Implicit in what we've said so far is the answer to my last question in **Part 1.**

The icosahedron, as we have seen, needs 3 colours, and truncation to give the football adds a 4th. But the dodecahedron already needs 4. Can truncation add a 5th? By the Four Colour Map Theorem, no. And you can see that, since only 3 faces meet in a vertex, and the faces produced by truncation do not touch, we can always choose a 4th colour to meet the requirement at each vertex.

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