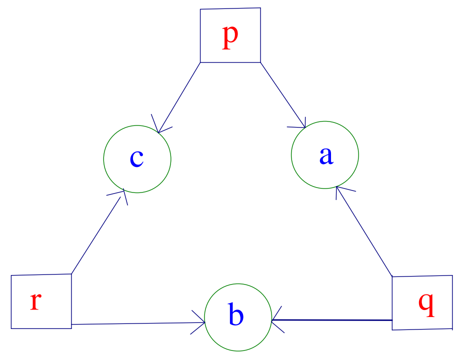
Magic Moments with Perfect Arithmogons

In formal mathematics we might organise the three equations

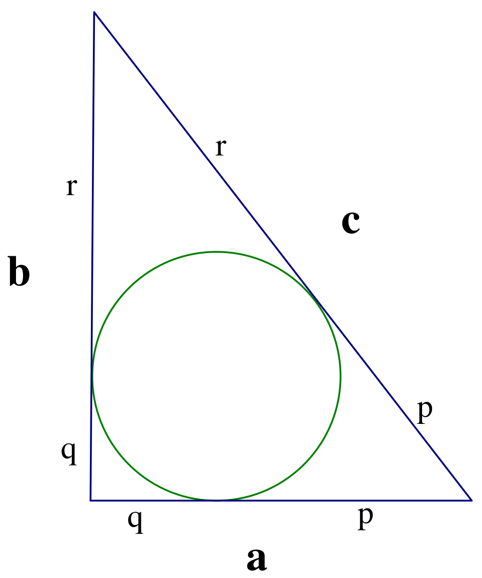
$$\begin{aligned} p+q \quad &= a \\ q+r &= b \\ p \quad +r &= c \end{aligned}$$

like this: $\begin{pmatrix} 110 \\ 011 \\ 101 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

In recreational maths we would do so like this:

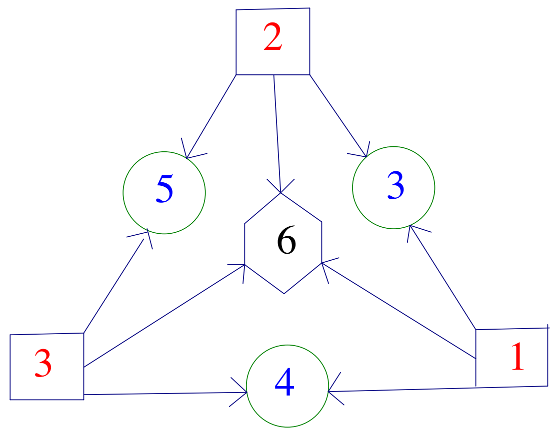


The triangular arrangement is called an *arithmogon*.



The triangle to the left has side lengths a = 3, b = 4, c = 5. Find p, q, r.

We soon pencil in the red numbers:



The arithmogon gives us an overview of the three separate additions. We're going to use it to show additions not just of pairs but of sets of 3, 4, 5 and more. You'll notice that we've added a central box summing the red numbers. Notice that the black number is the sum of a red number and the opposite blue number, and that the blue numbers sum to twice the red.

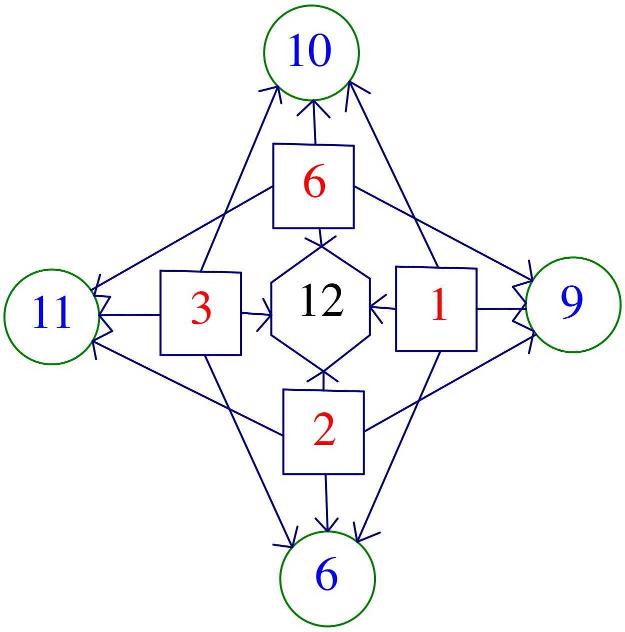
But I chose that particular example because of a special property. ***Each red number divides the sum of the others***, so the blue number opposite. Therefore it also divides the total of all three, the black number. We shall call such a set of red numbers, and the arithmogon representing it, *division perfect*, or just *perfect* for short. Our diagram is a perfect arithmogon of order 3, 3 because it contains 3 red numbers.

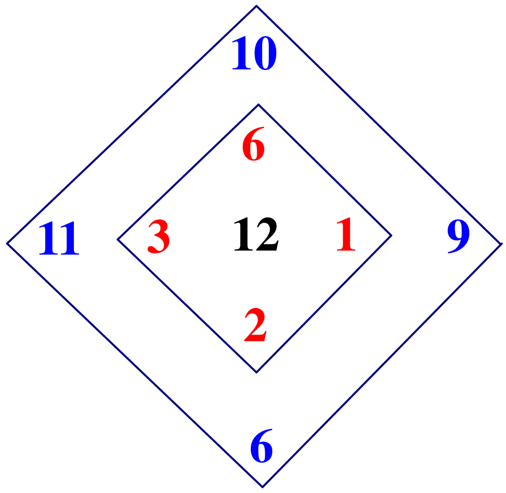
We shall allow no repeats, otherwise we could have sets consisting completely of '1s': $\{1,1,1\},\{1,1,1,1\}, \ldots$

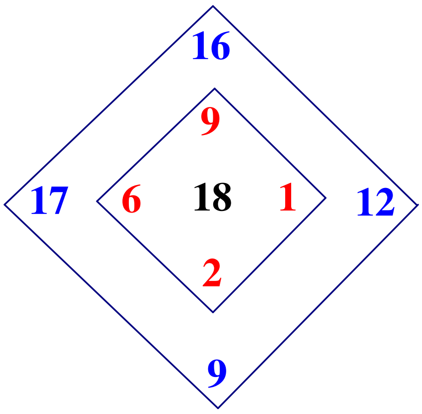
And we shall only allow *primitive* solutions, not those sharing a factor, so $\{1,2,3\}$ is fine but $\{2,4,6\},\{3,6,9\},\{4,8,12\},\ \ldots$ not.

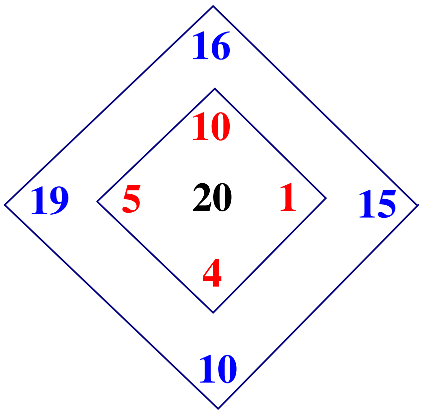
Here are some perfect arithmogons of higher order.

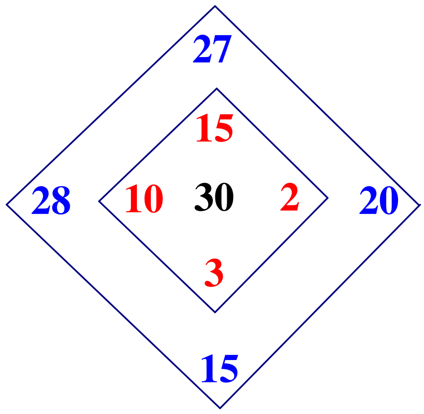
In this perfect order 4 arithmogon, as required, a red number complements the blue number opposite to give the black one. This time the blue numbers total ***3*** x the red. On the right we show it without arrows and boxes. To avoid a tangle of lines with higher orders, this is how we'll show arithmogons from here on.

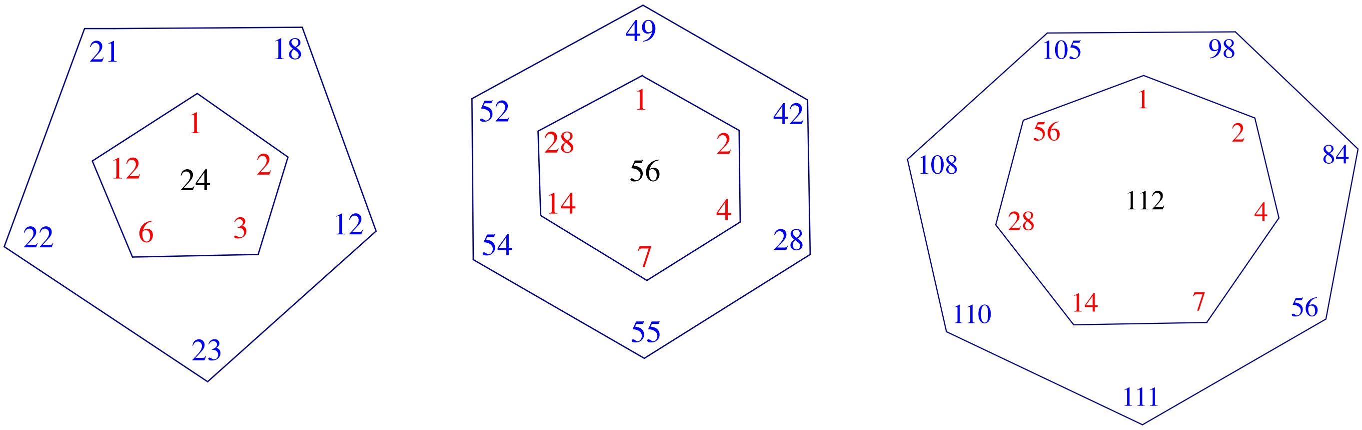












As you can confirm, we can generalise the observation we just made. The blue numbers sum to (n – 1) x the black number, where n is the order.

Some of the sets we've just pictured are included in this list:

Order	**Set**	**Total**	**Prime factors**	**Divisors of total missing from set**
3	1, 2, **3**, (**6**, **12**, **24**, …)	6	2, 3	
4	1, 2, 6, **9**, (**18**, **36**, **72**, …)	18	2, 3	3
	1, 4, **5**, **10**, (**20**, **40**, **80**, …)	20	2, 5	2
	2, 3, 10, **15**, (**30**, **60**, **120**, …)	30	2, 3, 5	5, 6
5	1, 2, 4, **7**, **14**, (**28**, **56**, **112**, …)	28	2, 7	
	1, 4, 10, **15**, **30**, (**60**, **120**, **240**, …)	60	2, 3, 5	2, 3, 5, 6, 12, 20
	1, 4, 20, **25**, **50**, (**100**, **200**, **400**, …)	100	2, 5	2, 5, 10, 20
6	1, 4, 40, **45**, **90**, **180**, (**360**, **720**, **1,440**, …)	360	2, 3, 5	2, 3, 5, 6, 8, 10, 12 15, 20, 24, 30, 60

The main thing to notice is that, as you move right, sooner or later you hit a number in **bold**. All the numbers to the right of it are also in bold. And each one is twice the last. These numbers sum all the numbers to the left of them. The numbers in brackets we can add in at will to make bigger sets. In some cases however, we have to do one or more doublings before we can complete the set (and enter the bracket). Two numbers are not only in bold but are picked out in green. If you look at the right hand column, you'll see that in these cases every divisor of the total is present in the set. These numbers, then, are the sum of all their divisors except the numbers themselves. The reason their divisors make up my perfect sets is that, if a number divides the total, it must divide the total less itself. Numbers like these are called *perfect* numbers. If you've read my piece *What makes perfect numbers perfect?,* you'll know a bit about these, but in any case I'll give the important facts here.

The formula for a perfect number is $2^{n-1}\left(2^n-1\right)$, where the number in the bracket is a prime of the *Mersenne* type. This gives you as divisors all the powers of 2 from 2^0 up to 2^{n-1} - there are n of those - ***plus*** all those numbers times the prime, except the last - that's another (n – 1). Therefore there are $n+(n-1)=2n$ – 1 in all. We had these examples:

$6 = 2(2^2 - 1)$. n = 2. Number of proper divisors: 2 x 2 – 1 = 3, giving us our order 3 arithmogon.
$28 = 2^2(2^3 - 1)$. n = 3. Number of proper divisors: 2 x 3 – 1 = 5, giving us one of our order 5 arithmogons.

Let's call these particular sets *doubly perfect*: perfect because they include all the divisors of the total, and perfect because each number divides the sum of the rest.

Euclid told us there's no biggest prime. And, though we can't be sure, there's no reason to suppose there's a biggest Mersenne prime either. If we used the biggest yet discovered to make a doubly perfect arithmogon which fitted round the Equator, one side would be no longer than the width of the laptop I'm typing on. n = 57,885,161 but N, the number of Mersenne primes known, is just 48.

As we've seen, a set of divisors of a perfect number is just one kind of division perfect set. Is there a general way to find all these sets? One way to make them is to find unit fractions which add to make 1. For example:

$$1 = \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{18}$$

Multiplying through by the LCM, which is 36, we have: 36 = 12 + 9 + 6 + 4 + 3 + 2 and each of {2, 3, 4, 6, 9, 12} divides the sum of the rest. Why is this? The pattern only emerges if we use algebra:

$$1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \ldots$$

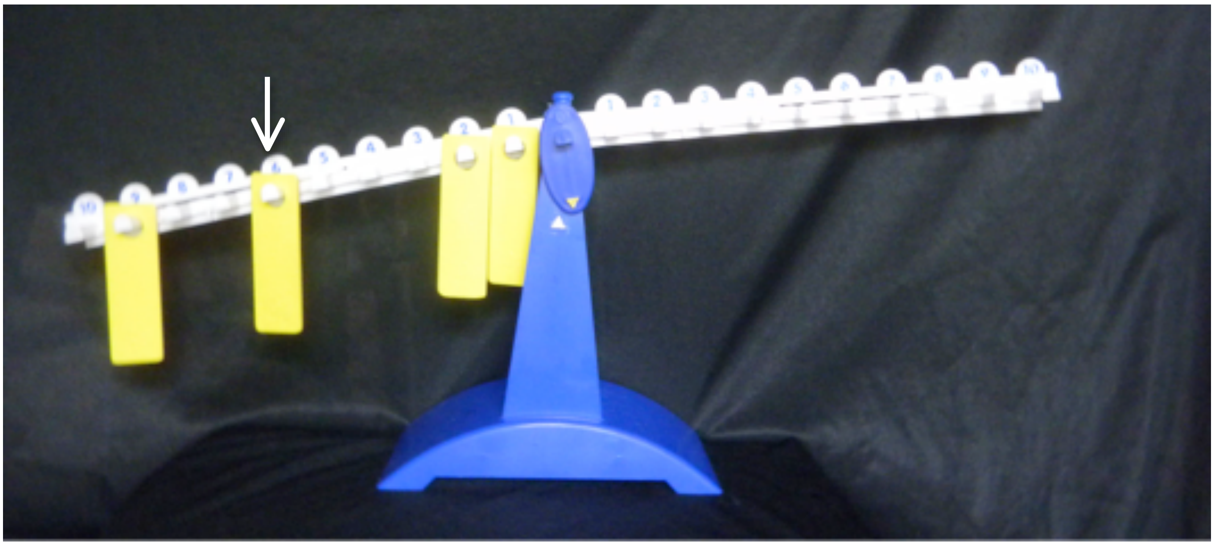
Multiply through by the LCM:
$(abc\ldots) = (bc\ldots) + (ac\ldots) + (ab\ldots) + \ldots$
Now subtract $(bc\ldots)$ from both sides:
$(abc\ldots) - (bc\ldots) = (ac\ldots) + (ab\ldots) + \ldots$
and factorise the left:
$(bc\ldots)(a - 1) = (ac\ldots) + (ab\ldots) + \ldots$

Since $(bc\ldots)$ divides the left side of the equation, it must divide the right. But the right is just the sum of all the other numbers.

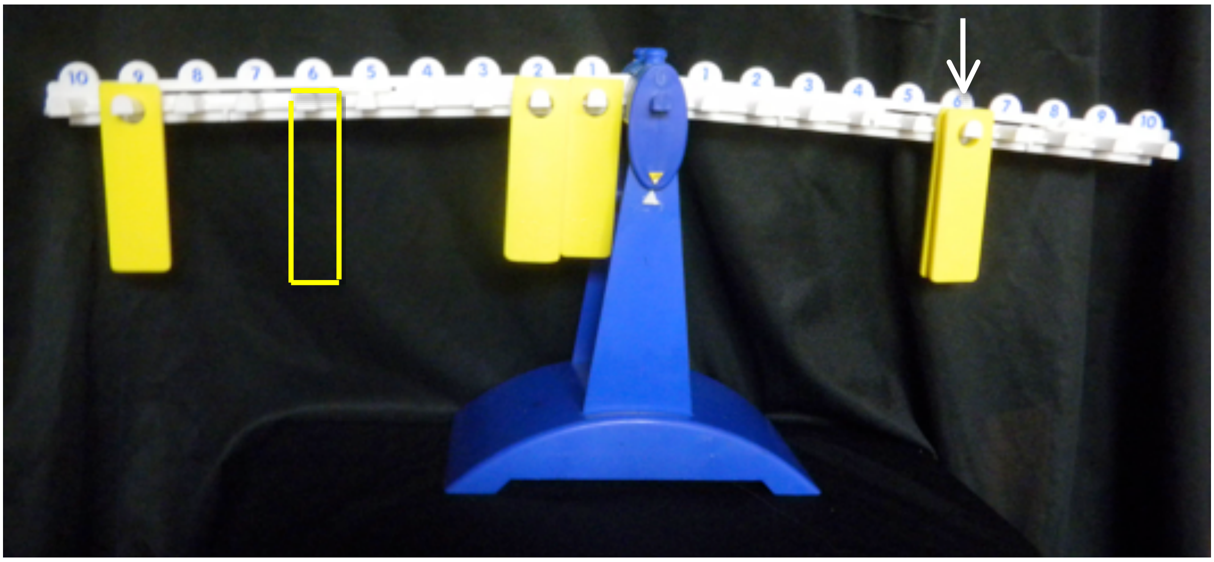
We can model these sets using a mathematical balance.

$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18}$ gives us 18 = 9 + 6 + 2 + 1 and the set {1, 2, 6, 9}.

We hang single masses from pegs 1, 2, 6, 9 on the left, then remove the '6' mass, say, …



… and hang it on the same peg on the right. All we need do to balance the beam is add the correct number of extra masses, in this case, 1:



This shows that $6 = \frac{1+2+9}{2}$ and we would find an equation of the same form for each mass.

Try other examples from our list on a balance.

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The Magic Mathworks Travelling Circus