LINES & PLANES
Experiments in projective geometry

As is usual in our workshops, we emphasise work which brings in the third dimension. The topic lends itself to this treatment since several of the results stated for two-dimensional configurations are more easily seen when the 3-D analogue is taken. We neglect what is arguably the most important concept in projective geometry: the duality of lines and points. Among other things, we do not discuss the diagrams as configurations.

Omit the stations concerning cross-ratio (**M9**, **E11**) with KS3 pupils. And, for a shorter session, or one with lower KS3 students, I suggest the following sequence, which omits a third of the activities. Confine the historical remarks to identifying the centuries in which the mathematicians worked and include only:

Part 1: **E1-5, 8, 10, 12-15**, i.e. only draw the grid of squares.
Part 2: E16-21, i.e. include only Desargues' and Monge's theorems.

However, for a general or adult audience, the historical notes are important.

Materials needed	*Activity/Comment*
Introduction	*Projective geometry is what the eye sees. When you look at a table, you don't see a rectangle, you see some strange quadrilateral. Your clever brain has to work out that it's rectangular.* *We're concerned with bunches of lines which radiate from a point. In this experimental work that point will be our eye. We'll use just one eye except at the end of **Part 1**, when we shall use a different colour for each eye and create the illusion of 3 dimensions.* *In projective geometry we don't need to measure anything; we just join points with straight lines. But there are interesting measurements we can make. An invariant is a mathematical quantity which doesn't change. We shall make measurements in **Part 1** to investigate a surprising example.*
	E1 *Teacher demonstration*
Perspex sheet, Dry-wipe pen, Pair of hinged perspex sheets	*We need to think about points, lines and planes. In this plane* [demonstrate with perspex sheet] *2 points define a line.* [Draw.] *Since the plane could rotate about this line* [demonstrate with perspex sheet]*, I need a 3rd point to define the plane. Those 3 points define a triangle, i.e. 3 lines, but I only need 2 of the lines to define the plane. Two planes meet in a line.* [Demonstrate.]
Skeleton pyramid, an oblique section shown with a rubber band	**E2** *Teacher demonstration*

If I look from the apex of this pyramid, I see a square. All the points in each of the four converging edges project as one. In mathematical language the mapping of points in a scene to points on the eye's retina is many-to-one. Thus the shape made by the rubber band, which projects as a square, could be anything. When you look at the constellation of the Great Bear in the night sky, you see the shape of a plough. But it just happens that the stars forming the vertices stand out because they lie in a dark part of the sky. If you could look from the side, you'd see that in fact the individual stars are at a range of distances from you. In that sense the constellation doesn't exist: it's an illusion. Consider people blind from birth who have their sight restored. Are they happy? No, they're not, or not for many years. They can make no sense of the new stimuli. When they walked around the three-dimensional world, they could make sense of it. But a world which maps many-to-one is, initially at any rate, incomprehensible. As infants it takes us some time to learn to distinguish from clues of apparent size, lighting and relative motion, what's near, what's far.

Part 1 Perspective drawing

Historical introduction: the work of Filippo Brunelleschi, Albrecht Dürer

E3 series

Dürer screen (D.S.),
Square,
Dry-wipe pen x 15

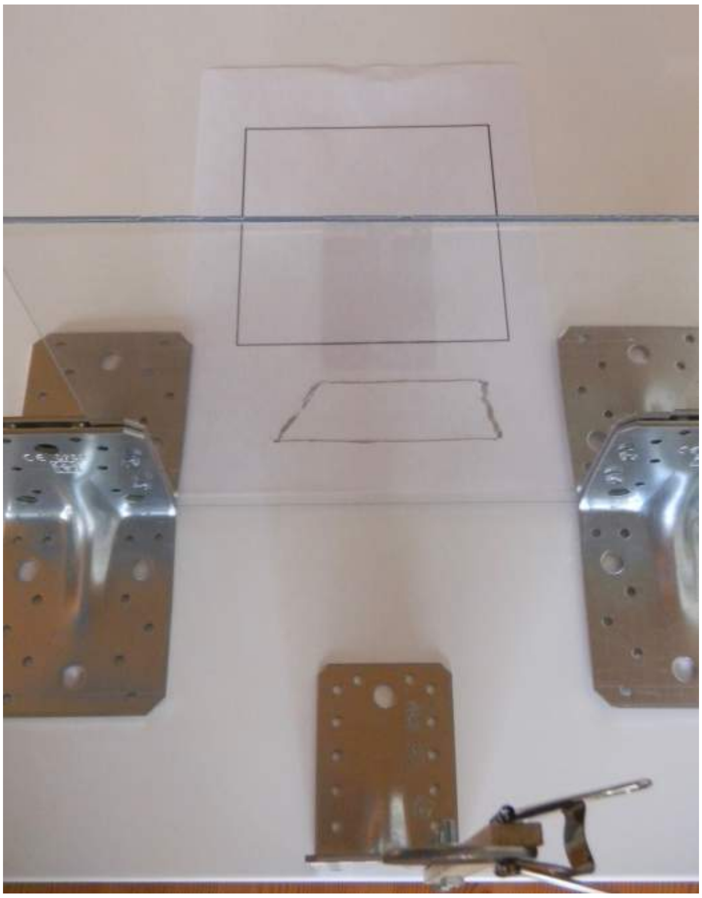
E3a *Class experiment*

Get your partner to hang the square vertically parallel to, and at increasing distances from, the screen. Trace the results. [As expected, the result is a

Series of (roughly) concentric squares.]

E3b *Class experiment*

Set the square square-on to the D.S.. Predict the image. Draw. Note the converging parallels, (resulting in a trapezium).



E3c *Class experiment*

Torch x 15

Reverse the ray path by shining a torch through the sight.

E3d *Class experiment*

Move the front edge of the square till it coincides with the back edge in the first position. Repeat till you have part of a railway track. Note that the parallel lines come closer and closer together.

E3e *Teacher demonstration*

Infinite plain/plane figure

Project.

Imagine that you're looking down on a railway track that goes off across an infinite plain. The horizon you see is not due to the curvature of the Earth. We call it 'the line at infinity' and the vanishing point 'the point at infinity'. We can't avoid the concept of infinity in mathematics but, wherever it appears, and however rigorously we can prove a result about it, it always retains an aura of mystery. On the right, you're looking further and further into the distance but the red angle never becomes zero. On the left you've taken a photograph framed by the red square. Below is an enlargement. You can keep on doing that for ever and it will always look exactly the same.

E3f

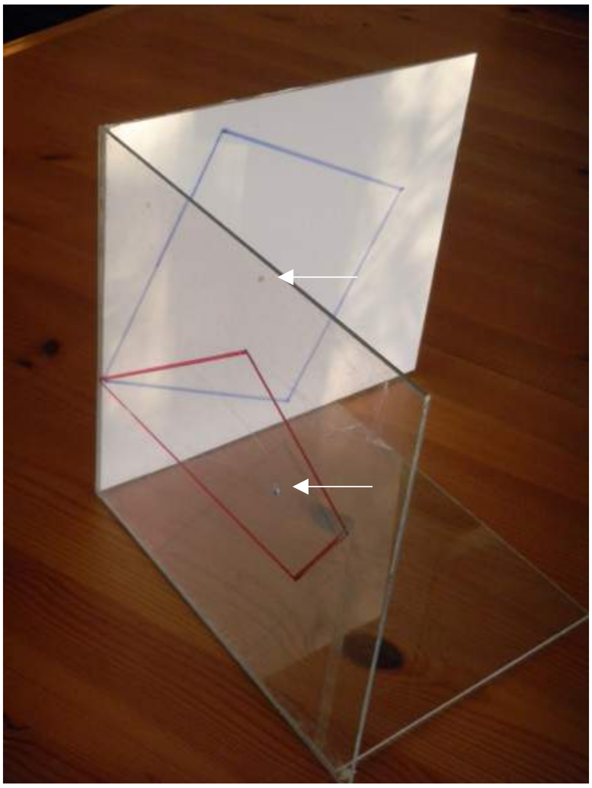
What do you think will happen when you set the square at 45° to the screen?
[You will obtain a kite.]

E4 *Class experiment*

Any convex quadrilateral can be projected as a square. [Recall **E2**.] Here's an example:

Model x 15

Sight up the holes. Move your eye slowly inwards. There will come a point where the red quadrilateral fits the blue square behind.



D.S.,
Equilateral triangle, x 15

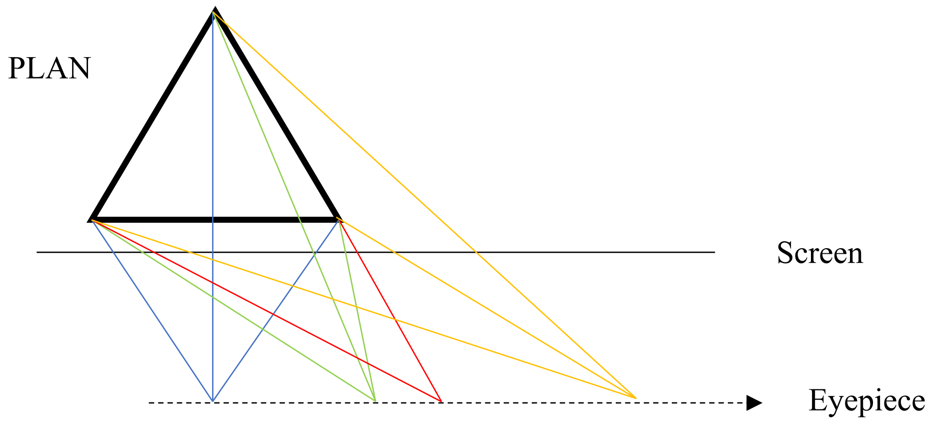
E5 series

E5a *Class experiment*

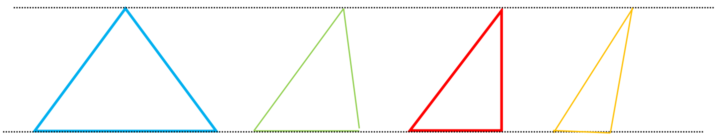
Set the triangle with base parallel to the screen. Try drawing with the eye-piece at different heights. [Image isosceles, increasingly obtuse as height falls.]

E5b *Class experiment*

What image will result if you move the eye-piece to the right? [Triangle loses symmetry axis, right base angle increases at the expense of both the apical angle and the left base angle, eventually becoming obtuse. The base shortens progressively. Sequence shown beneath.]



ELEVATION

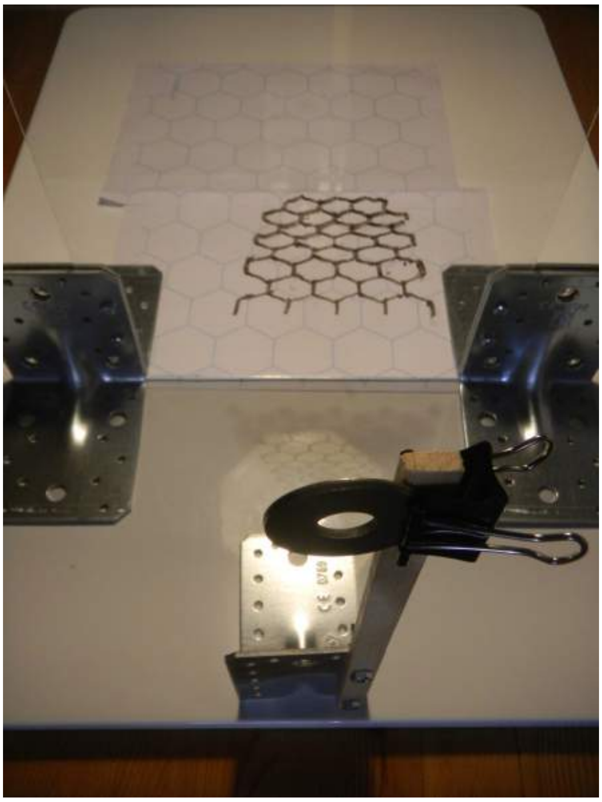


The images overlap but have been separated for clarity.

E6 *Class experiment*

D.S.,
Hexagon tiling (A) x 15

Draw a part.



E7 *Class experiment*

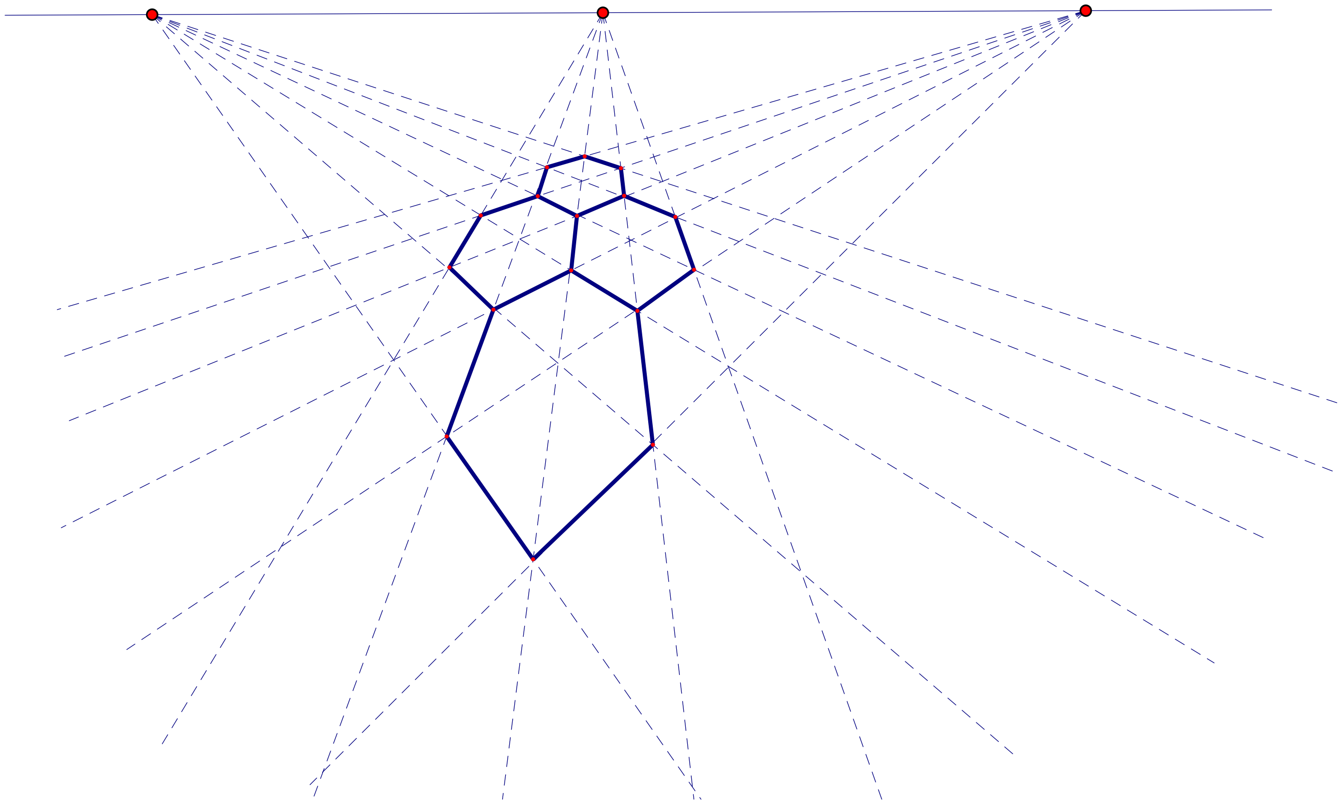
Paper,
Pencil,
Ruler x 15

How do we draw this without the aid of the D.S. ?

Teacher demonstration

Computer with GSP sketch of (A)

Project.

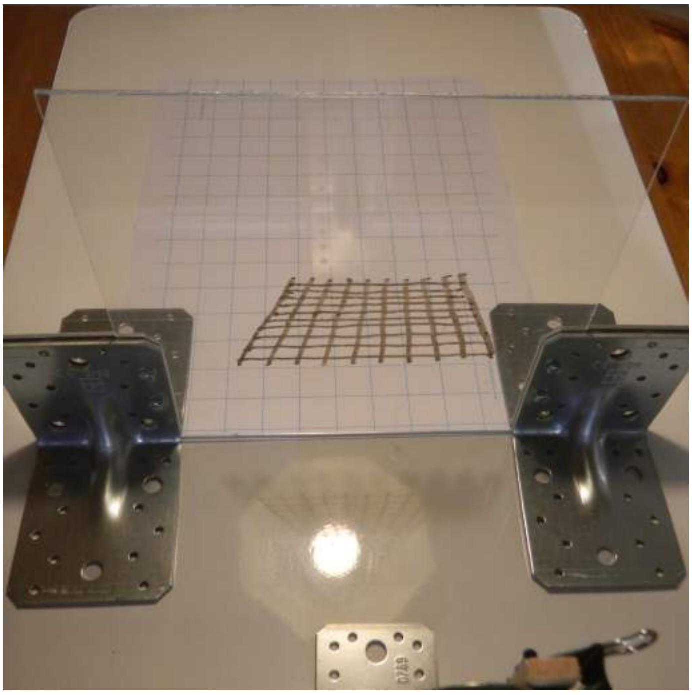


(A) *Each set of parallels has its own vanishing point. There are 3 sets of parallel diagonals, on which the 3 pairs of sides also lie.* Draw a horizon and choose the 3 vanishing points. Use a different colour for each. Draw part of the tiling.

E8 *Class experiment*

D.S.,
Checkered tiling (A)

Draw a part. Again notice parallels converging to a 'vanishing point' on the horizon. Notice how the apparent depth of the tiles shrinks with distance.



M9 *Teacher demonstration*

Blank acetate on OHP,
Dry-wipe pen

What is the shrinking rule?

Define and illustrate cross-ratio for a single pencil of 4 lines.
Show the principle and structure of the 'triangle area' proof of invariance.

Explain that all pencils from infinity come from the same point so have the same cross-ratio.

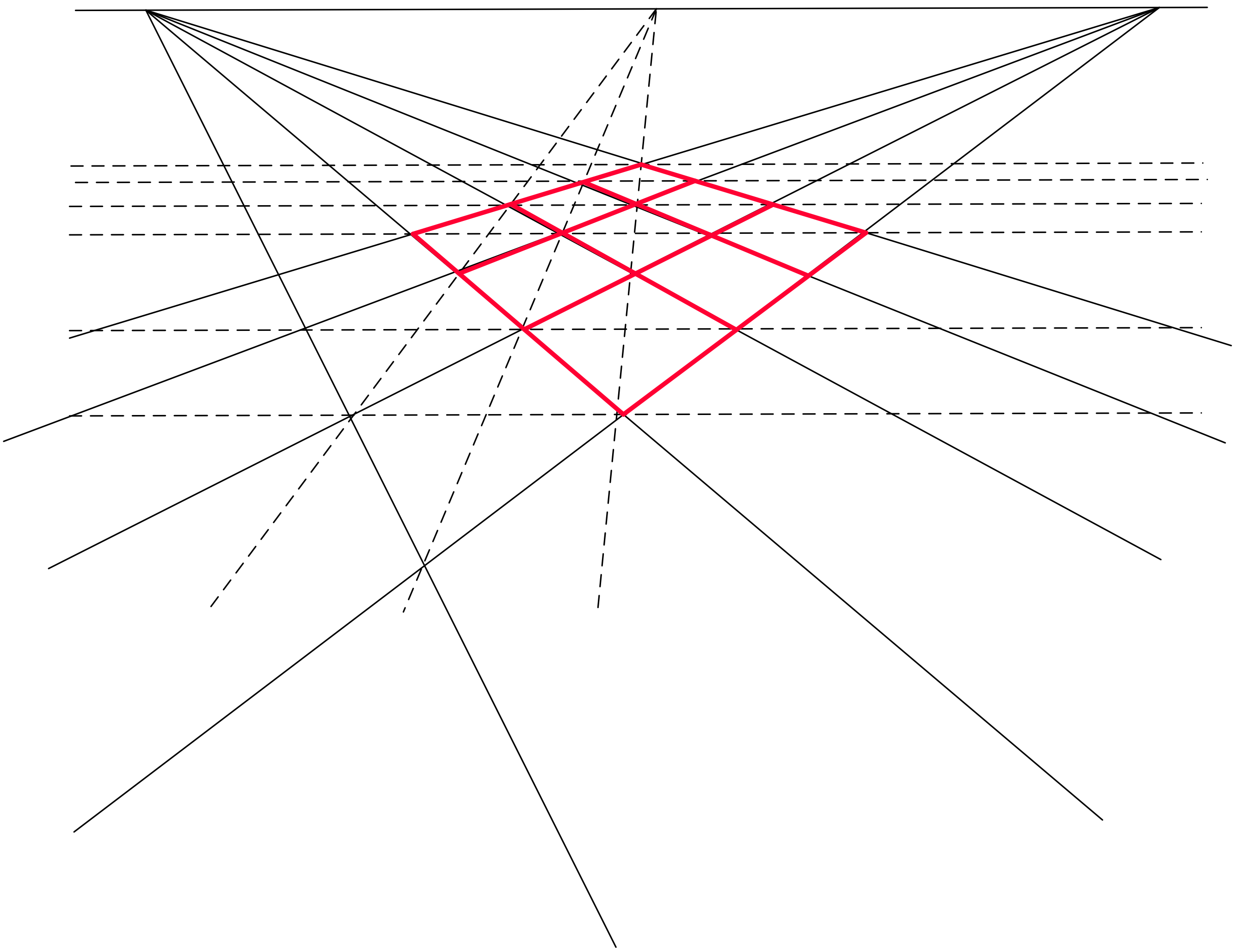
Explain that the experimental check will be deferred to **E11**.

E10 *Teacher demonstration*

Again, how do we draw this without the aid of the D.S.?

Computer with GSP sketch of (B)

Project.



Class experiment

(B) *There are 2 sets of parallel diagonals and 2 sets of parallel sides. It is convenient to have 1 diagonal parallel to the imaginary drawing surface so that the lines run off to infinity in either direction.*

Draw a horizon line and choose 2 only of the vanishing points. Having got one intersection, draw a horizontal diagonal through it. Make the next pair of lines from the vanishing points intersect on it. Proceed in this way. Trace the other set of diagonals and they will meet in a third vanishing point midway between the other two.

The alternative is to set the three, differently coloured, vanishing points at equal spacings along the horizon line. Colour-code the lines from them and simply observe the rule that lines of all three colours must pass through each intersection.

Draw part of the tiling.

E11 *Teacher demonstration*

Computer with GSP sketch of (B) again but with measurements.

Project. Show measurements and cross-ratio results.

E12 *Class experiment*

D.S.,
Dry-wipe pen,
+ ff. x 15:

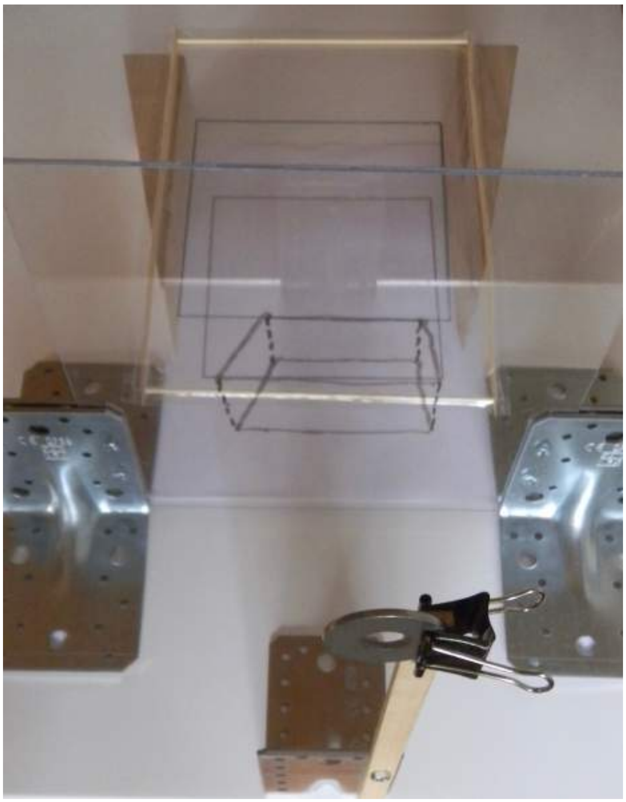
Acetate square laid above original to form square prism

There are now 3 sets of parallel edges, each with its own vanishing point. Convergence of vertical lines is just noticeable here but is extemely noticeable in photographs of high buildings. (The brain tends to correct for this.)

Architectural photograph

Project.

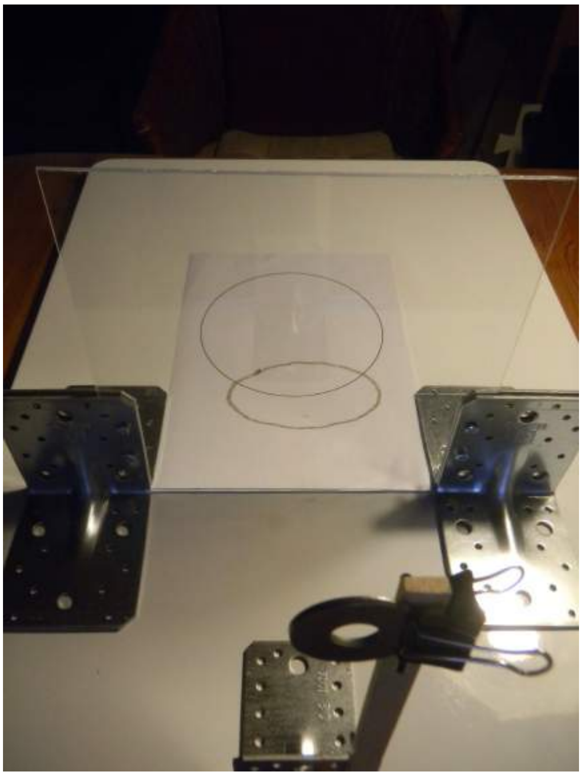
Draw. *Artists call the apparent vertical squashing of parts of an object in the projected image 'foreshortening'. Note the greater foreshortening in the layer near eye level.*



E13 *Class experiment*

Circle

Draw. Lower the sight and repeat. *Note how the ellipses become more squashed – their 'aspect ratio' (length : width) increases, i.e. there's greater foreshortening.*

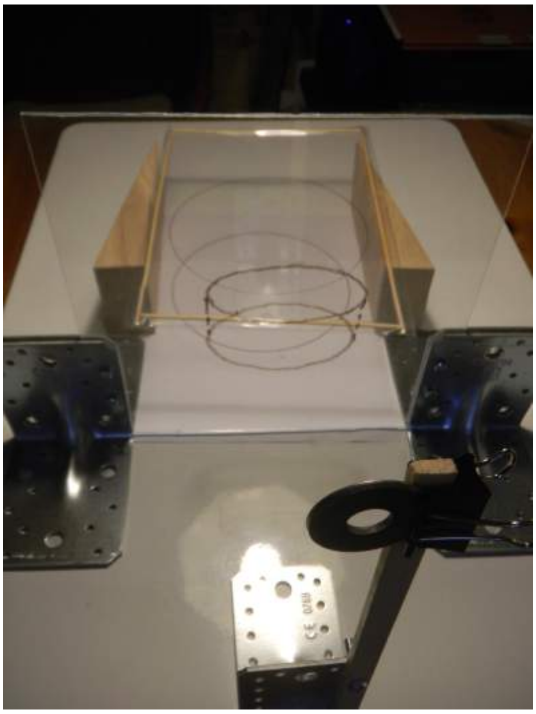


E14 *Class experiment*

That effect will be clear in this experiment.

Acetate circle laid
above original
to form cylinder

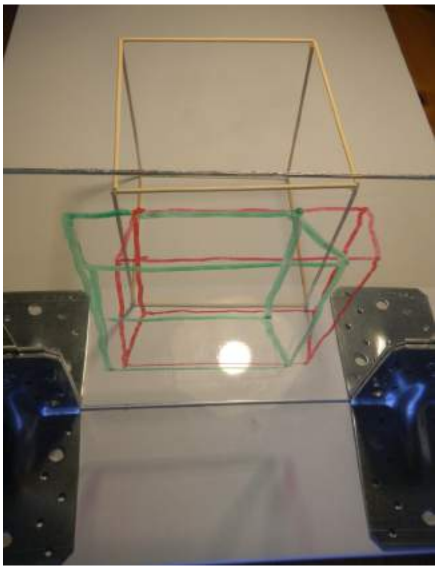
Draw.

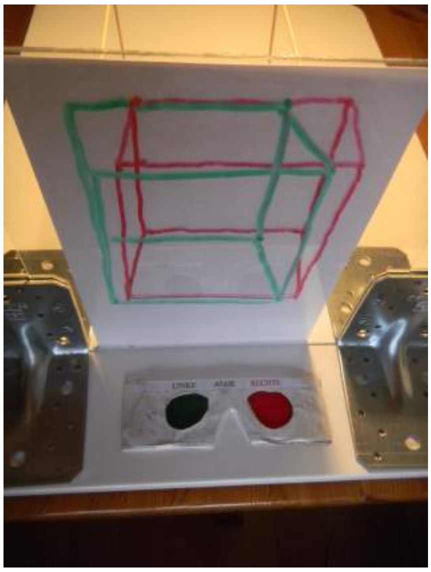


E15 Class experiment

D.S., Skeleton cube,
Red, green dry-wipe pens,
Anaglyptic spectacles

Shut your left eye and draw the cube with the red pen.
Shut your right eye and draw the cube with the green pen.
Place a white card behind the screen.
Put on the red (right)/green (left) anaglyptic spectacles.
You should see a black cube in three dimensions.





Part 2 Surprising results.

(a) Three theorems and their surprising proofs by moving from 2 dimensions to 3.

We are now going to look at some figures which we draw in an arbitrary way and a straight line magically appears. We shall seek an explanation by imagining that the figure in cases (i) and (iii) is the projected shadow of, and in case (ii) a section through, a three-dimensional structure,

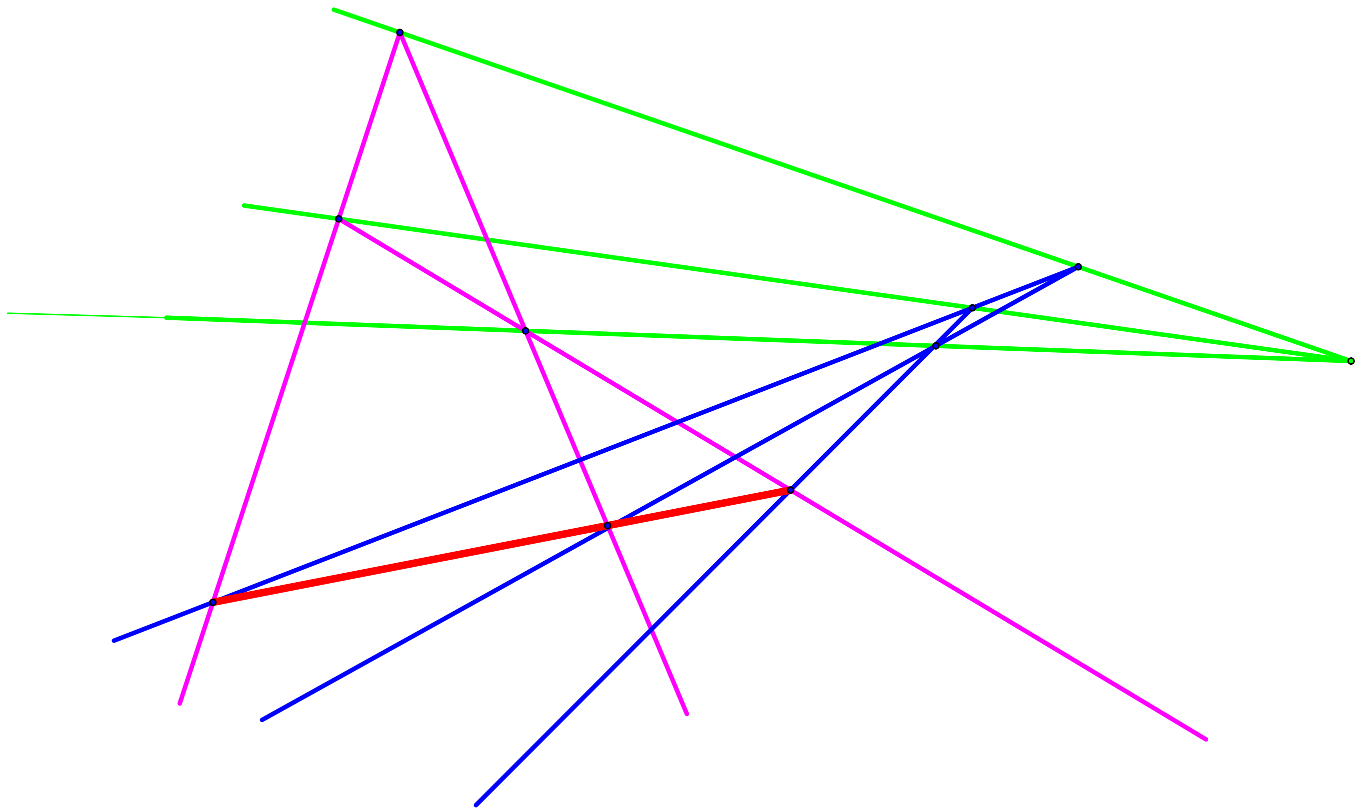
(i) Désargues' theorem

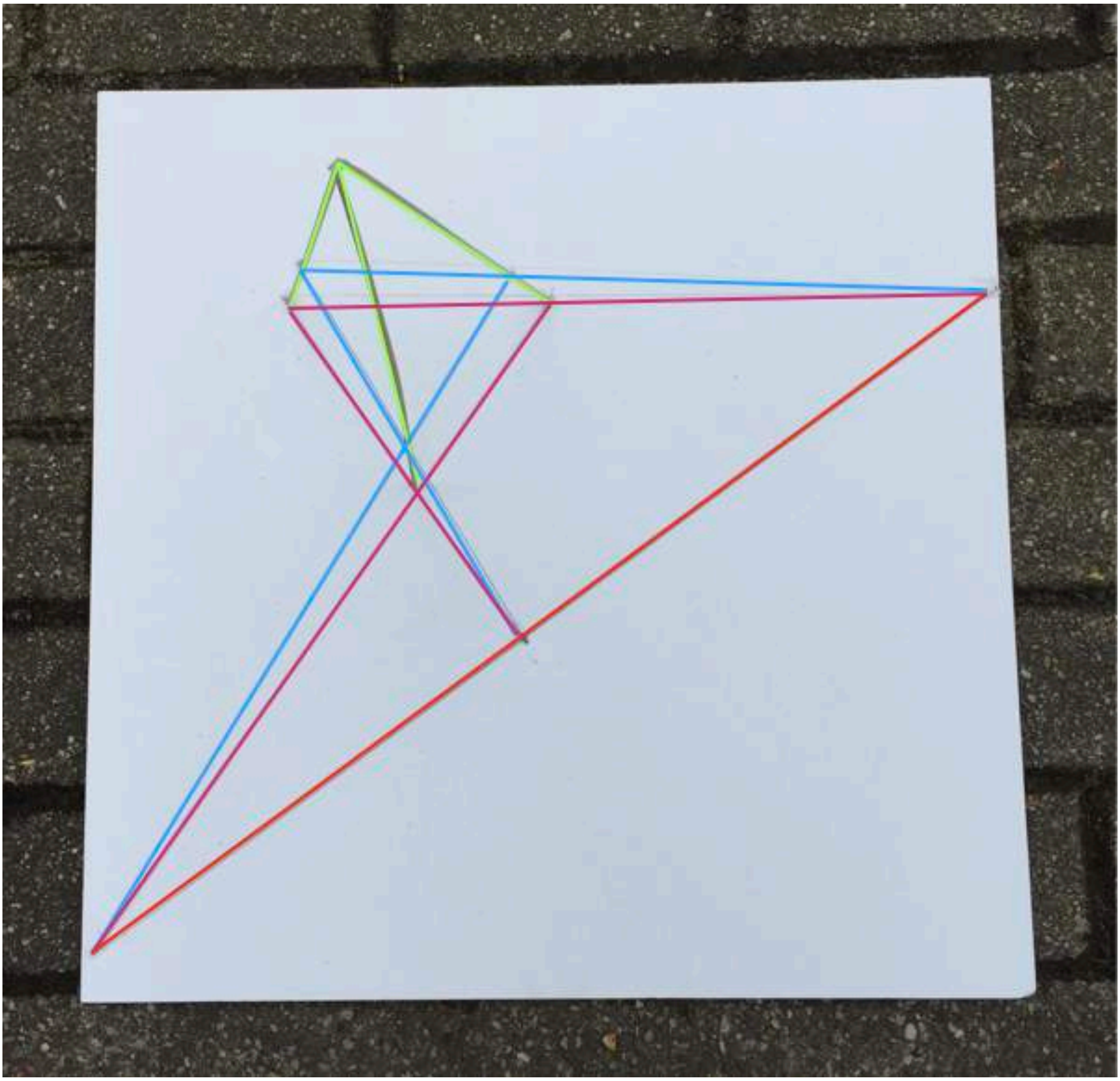
E16 *Teacher demonstration*

Computer with GSP sketch — Project. Use the .gsp file to move the points around.

E17 *Teacher demonstration*

3-D model — Exhibit.



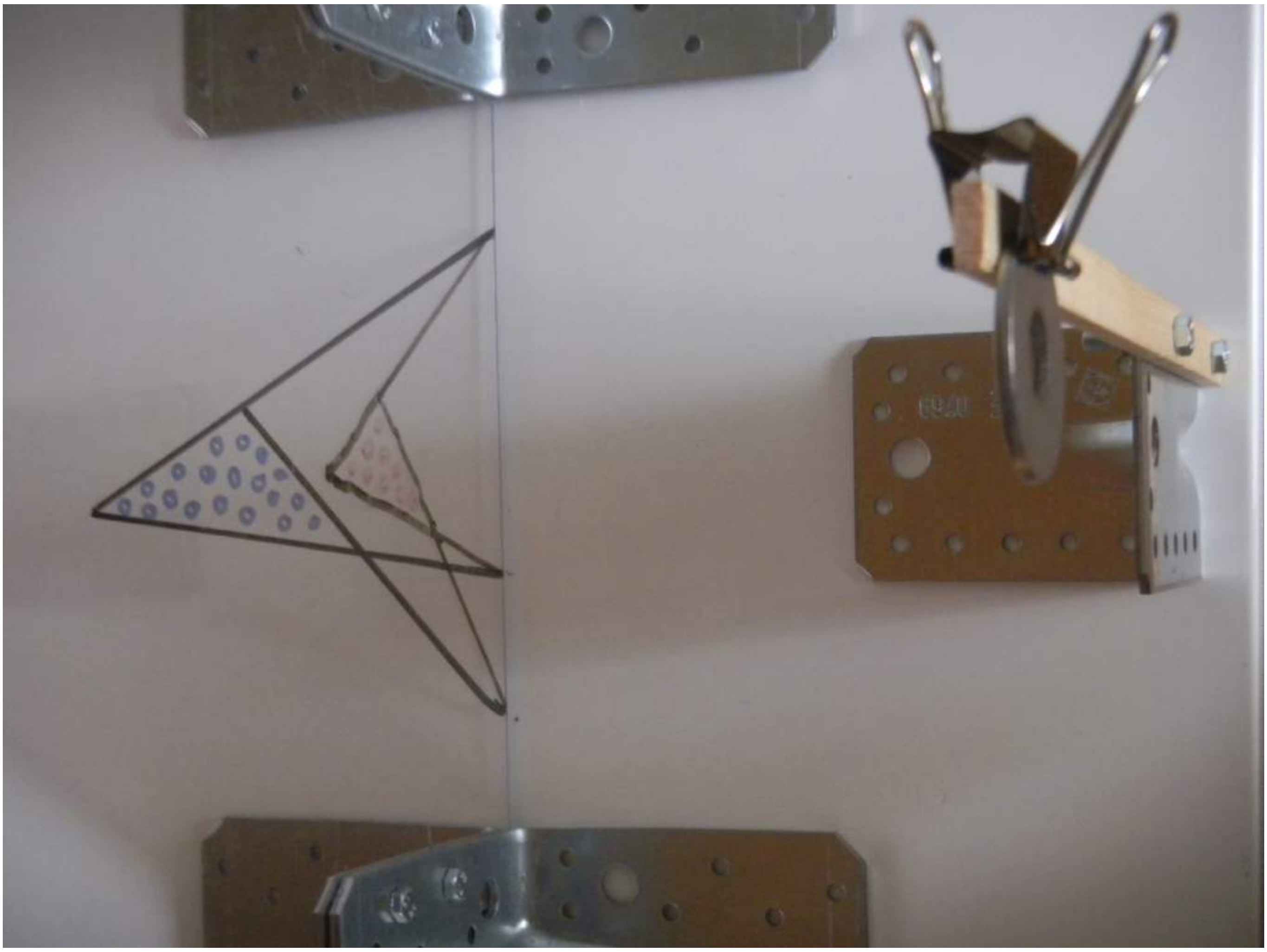


We've lifted the point from which the 3 lines come up into the air.[Indicate.] *The blue points and lines now define a sloping plane. The purple points and lines define the plane of the white board. And we know that two planes meet in a line.*

D.S.,
Dry-wipe pen,
Triangle, x 15

E18 *Class experiment*

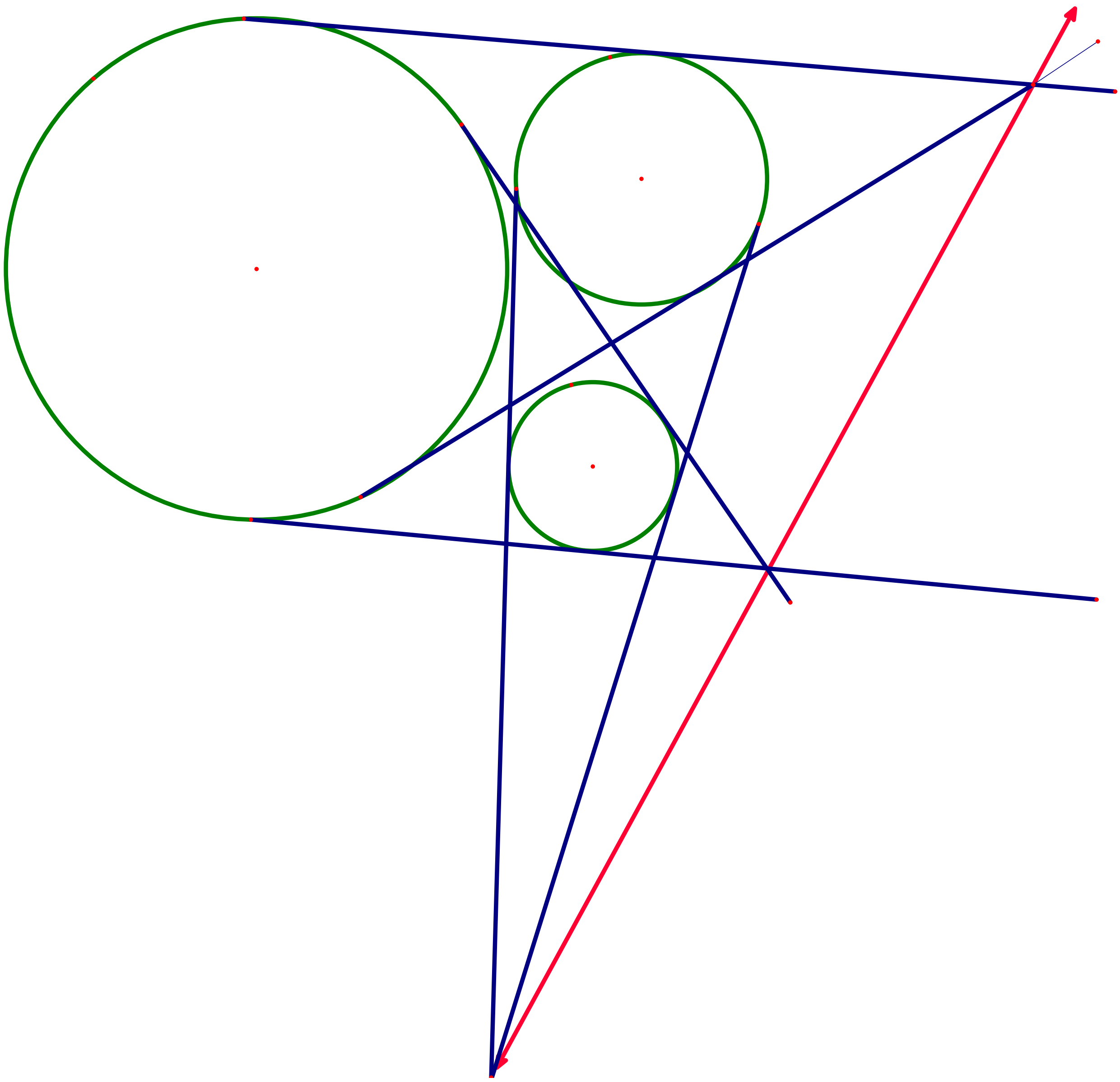
One person: Draw a triangle on the baseboard behind the screen.
Colleague: Draw it on the screen as before.
What are you expecting to happen when you extend the sides of the two triangles to the junction of screen and table? [The lines will meet in pairs there.]



(ii) Monge's theorem

E19 *Teacher demonstration*

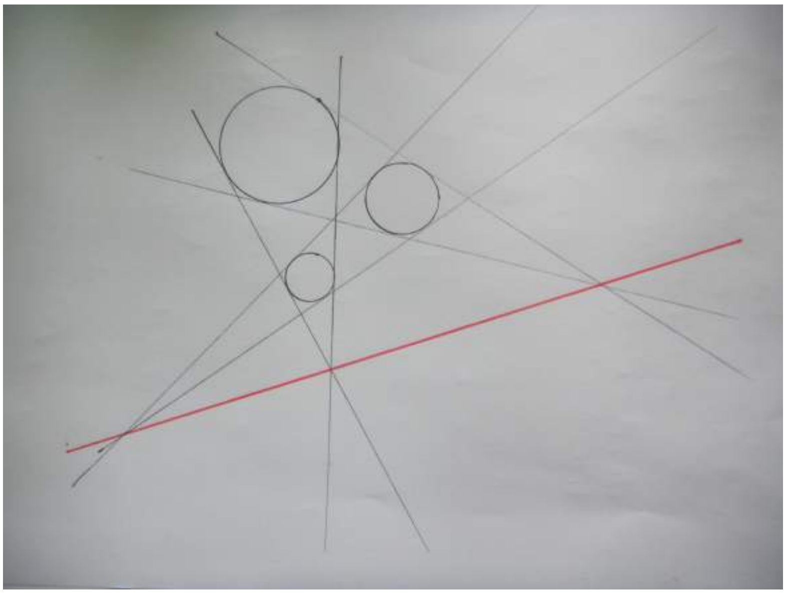
Computer with GSP sketch Project.



Class experiment

Paper,
Pencil,
Circle stencil,
Straight edge

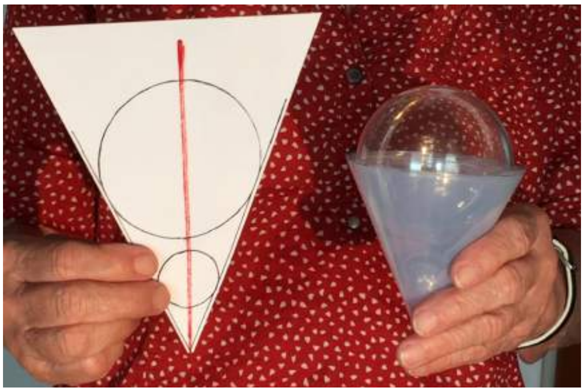
Make your own drawing.



White magnetboard, Perpex hemispheres of 3 sizes with embedded magnets

E20 *Teacher demonstration*

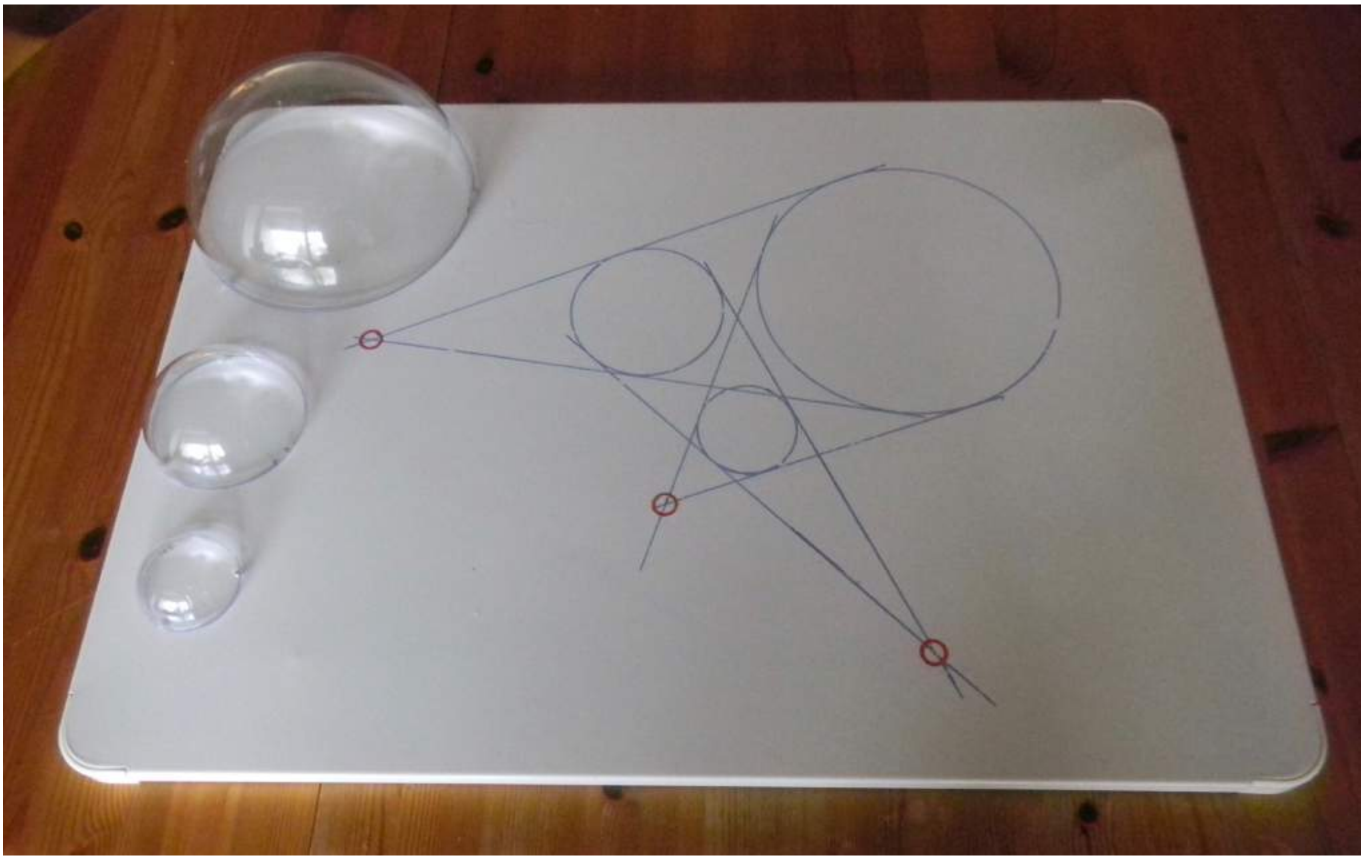
As a preliminary, present the card on the left below. Ask what you get if you rotate it about the red axis. When the children have replied, present the model on the right alongside it.



Perspex sheet, Drywipe pens

The magnetboard should be set horizontally but at the centre of the class so that all can see it.

1.1 Arrange the 3 hemispheres arbitrarily on the board.
1.2 Draw round the hemispheres then remove them.
1.3 Draw the external tangents to each circle pair.

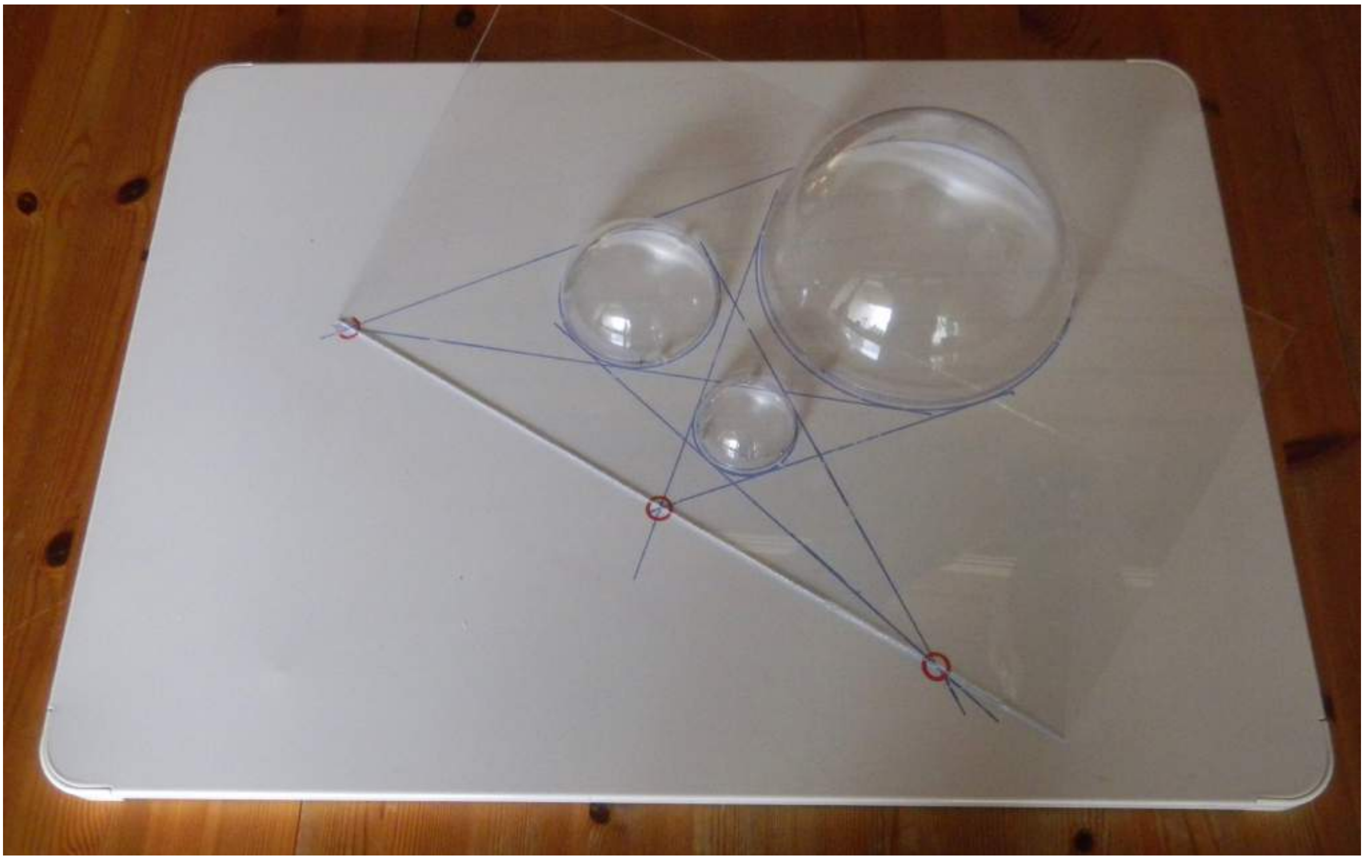


2.1 Replace the hemispheres.

Having described stage 2.2, ask the class what they think will happen.

2.2 Maintaining contact with all 3 hemispheres, slide the perspex sheet till it meets the board.

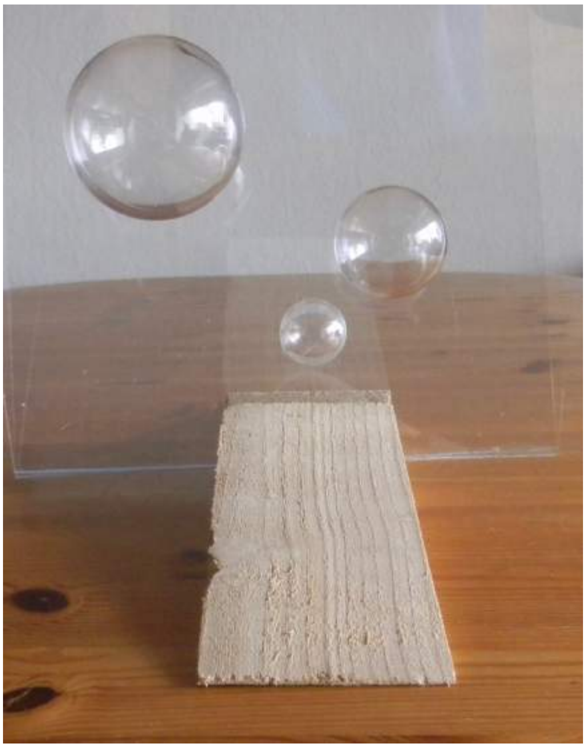
[It will meet the board in the line of intersections.]



ff. x 15:
Perspex ‘book’,
Spheres of 3 sizes,
2 stands to support
book in a ‘V’,
Dry-wipe pen

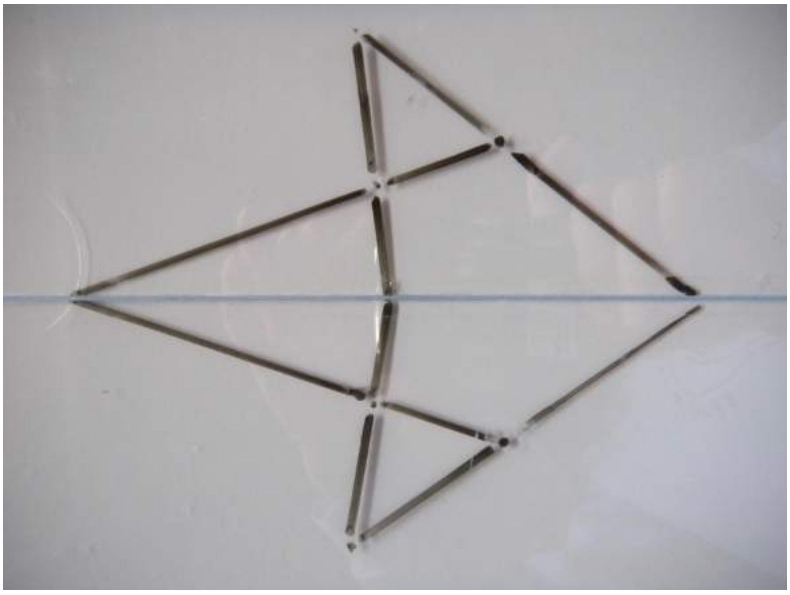
E21 *Class experiment*

Drop the spheres into the ‘V’: each will find its own level.

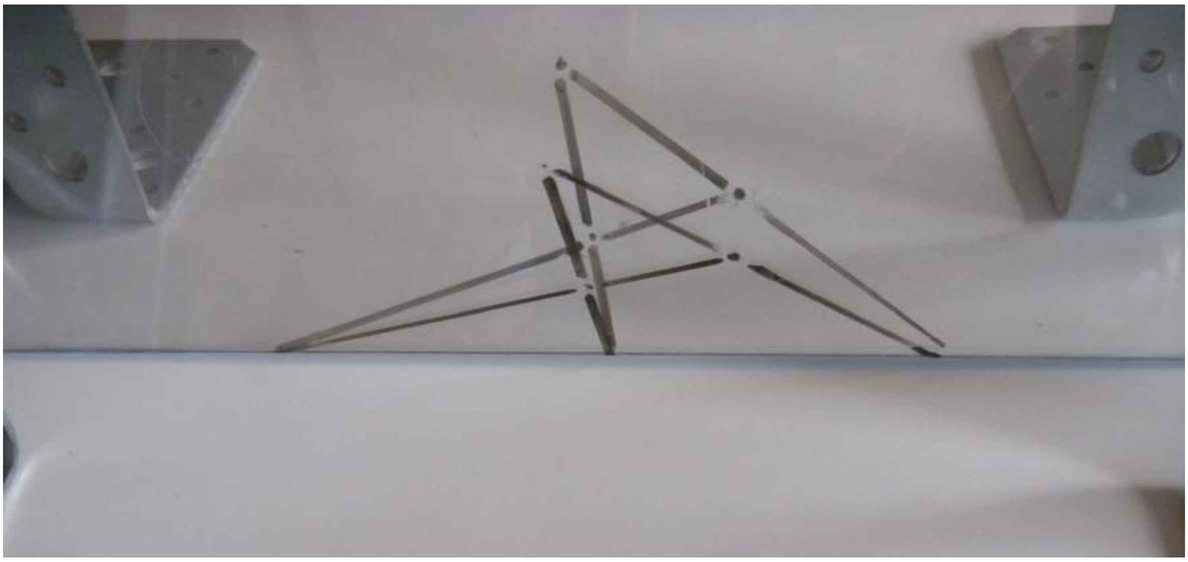


Because the spheres are reflected in the perspex, their points of contact with the 'book' pages can be marked with fair accuracy:

Flatten out the 'book' and join each pair of points to the hinge. Because we had a vertical symmetry plane down the centre of the V, we get a symmetrical two-dimensional figure. [Exhibit.]



Refolding the 'book', we see the positions of the lines in three dimensions:

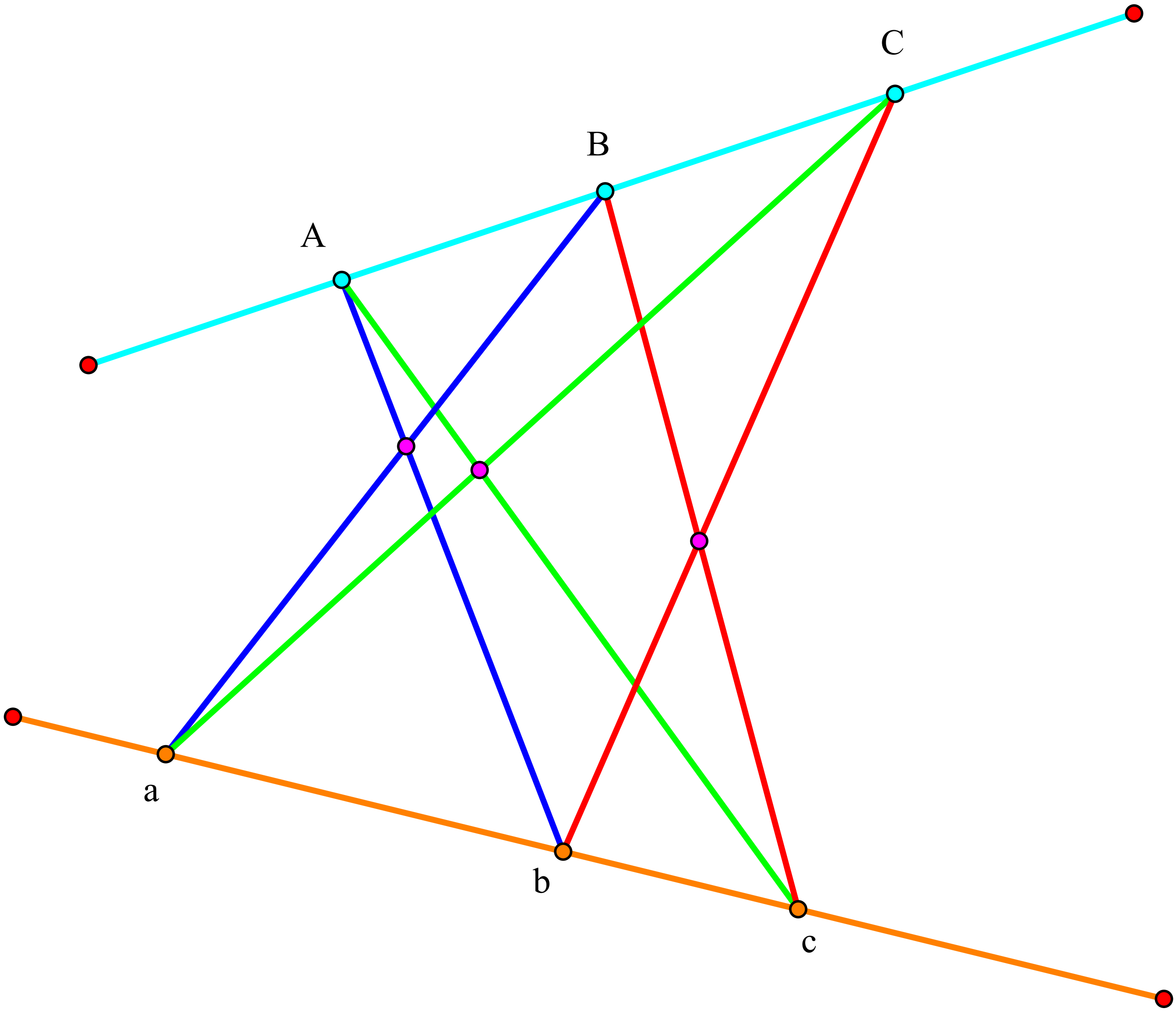


*Notice that this 3-dimensional figure is the same as that for the 3-dimensional figure for Desargues' theorem. (Look back at the model for **E17** and the two triangles in **E18**.) The 3 sticks of **E17** would be parallel and the point where they meet therefore an infinite distance away in both directions.)*

(iii) Pappus' theorem

E22 *Teacher demonstration*

Computer with GSP sketch — Project. Use the .gsp file to move the points around.



Class experiment

Paper, pencil,
Straight edge

Make your own drawings.

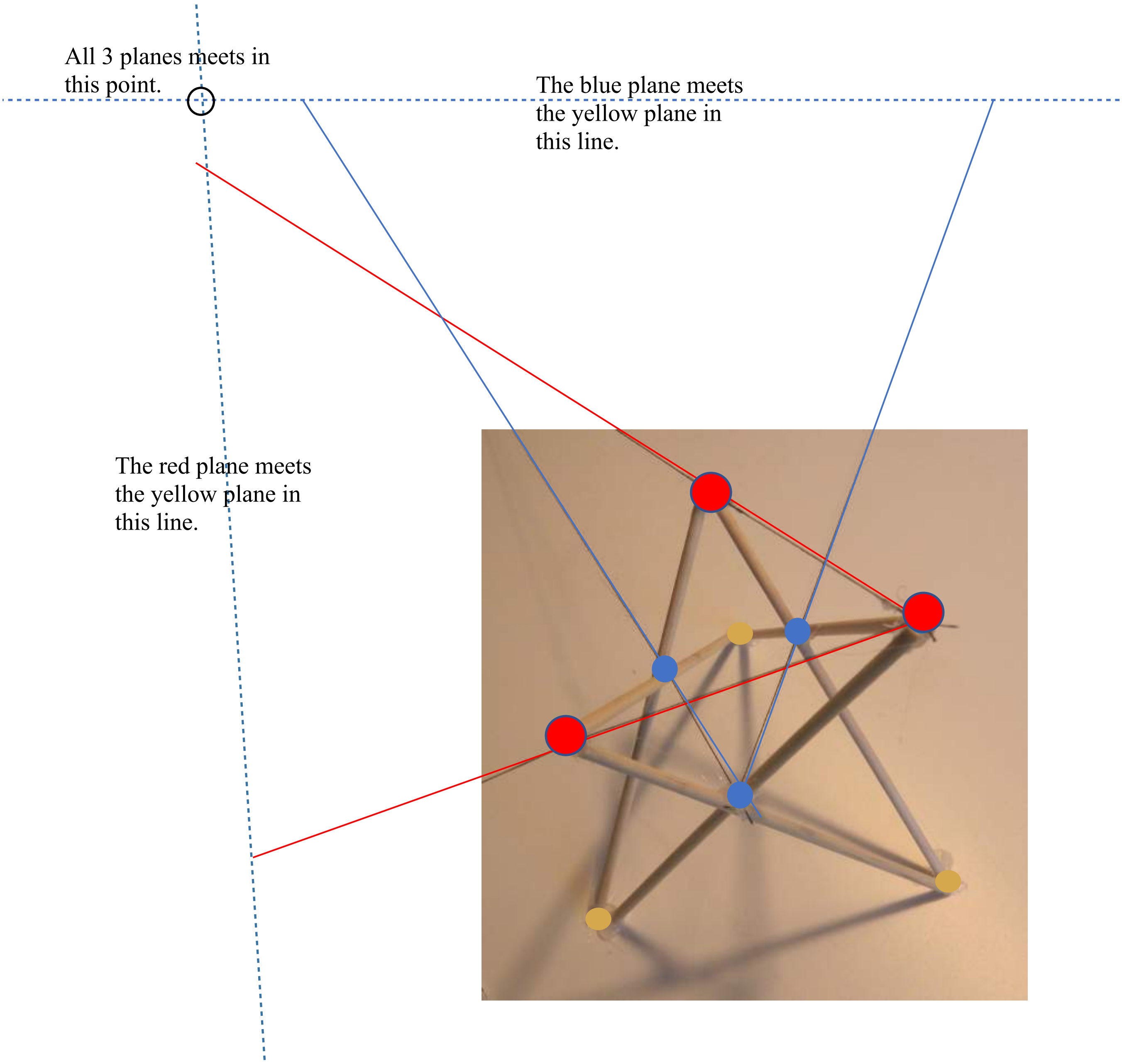
E23 *Teacher demonstration*

3-D models (A, B)

Exhibit.

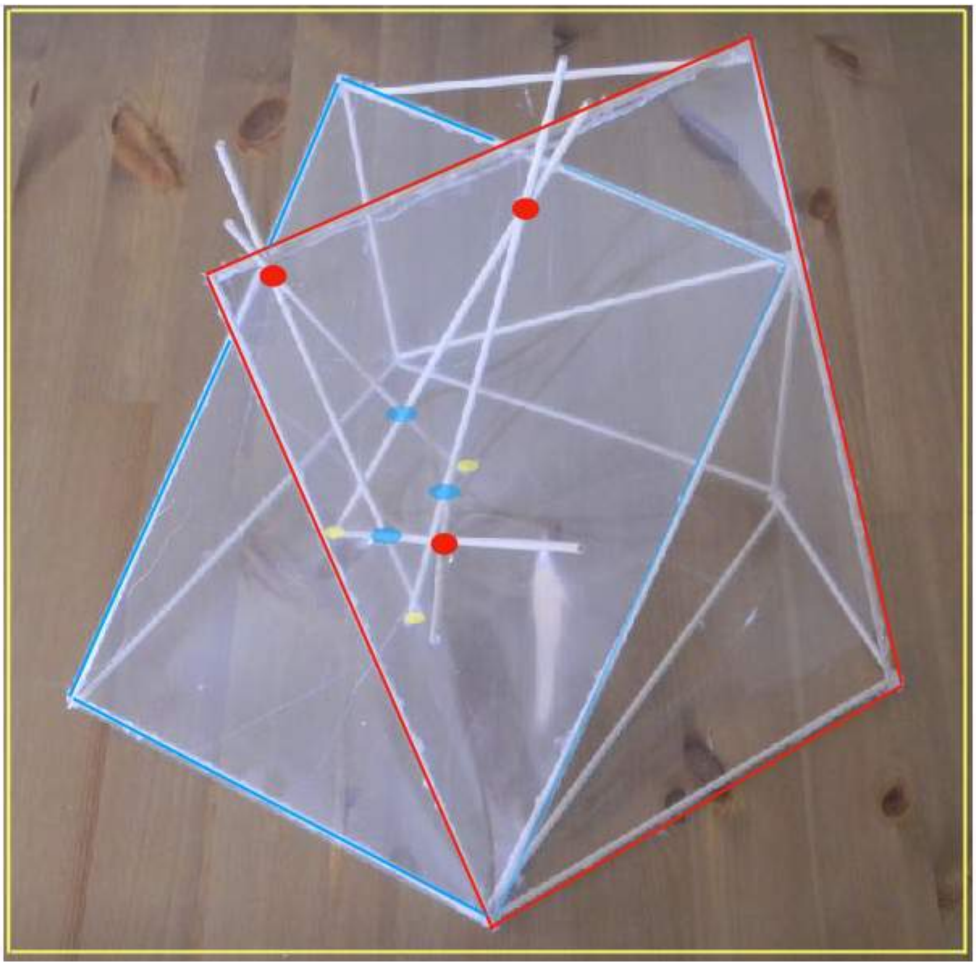
Model A:

Since point O lies in all 3 planes, seen from here, the 3 red points form a line, the 3 blue points form a line and the 3 green points form a line.



Model B:

Here the planes are shown. The point O is the bottom vertex.



(b) A surprising generalisation of Pappus' theorem ((a) (iii)): Pascal's theorem

E24 *Teacher* d

emonstration

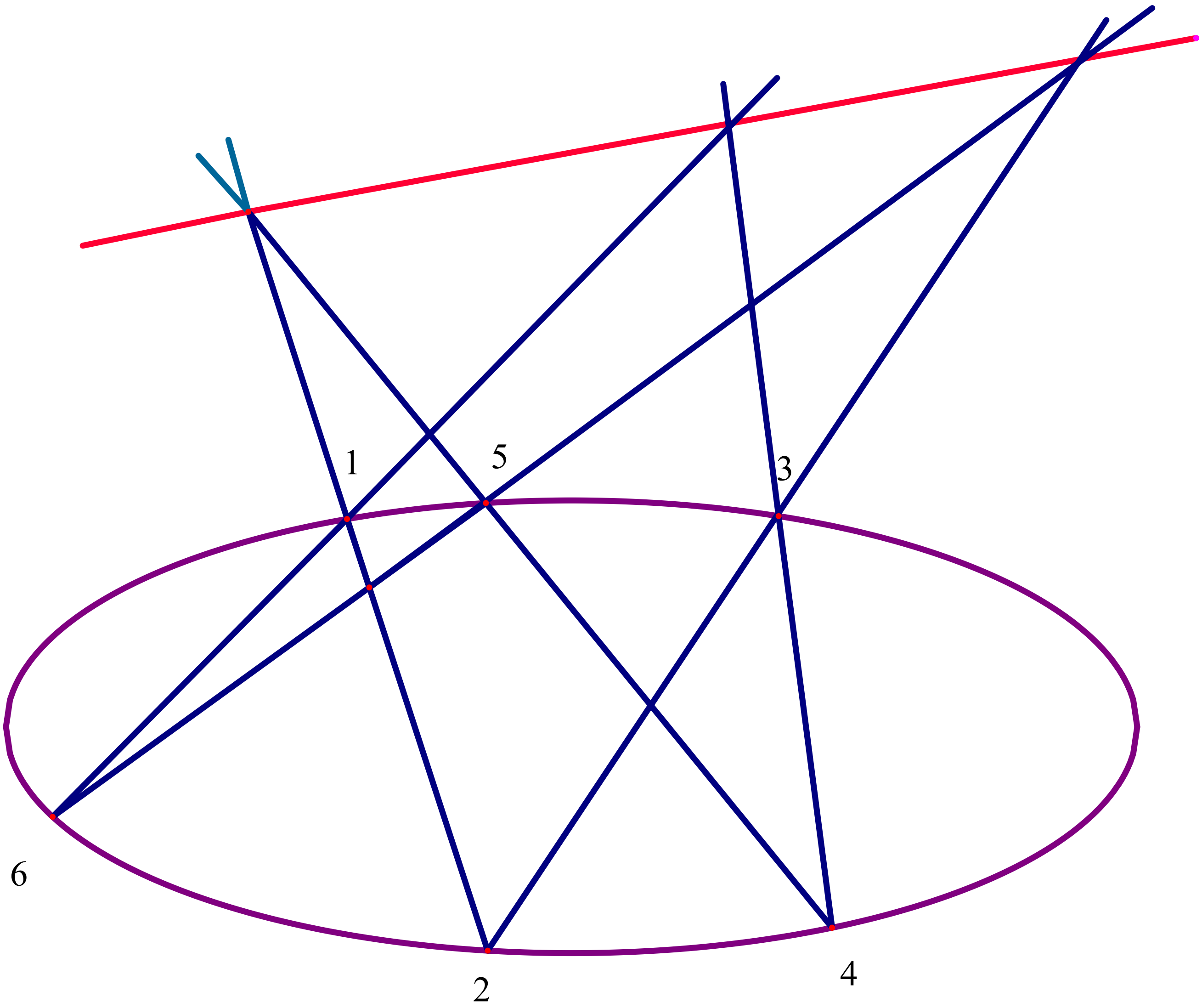
Acetate on OHP

We shall not do the geometry, which depends on elementary circle properties, but we can show that the Pappus property applies not just to a pair of straight lines but to a circle.

And that's not all.

Set the projector at an angle to the screen.

When we project a figure, straight lines stay straight lines but a circle may turn into an ellipse as here.



E25 *Teacher demonstration*

Slit on OHP, sectioning double cone

Circles and ellipses are sections of a cone.

What we have just shown is Pascal's theorem, which generalises Pappus' theorem to any conic section– you can think of the two Pappus lines as the sides of an infinitely thin ellipse. It was discovering this which convinced the philosopher Blaise Pascal that henceforth he should devote himself to the study of mathematics.

(c) A surprising freedom: Poncelet's 'closure' theorem.

I want to draw a triangle which fits between two circles, that is to say, it is tangent to the inner one and circumscribed by the outer.
No problem. There is a simple formula into which I fit the radius of the smaller circle, the radius of the bigger, and it tells me how far apart to set their centres.

E26 *Teacher demonstration*

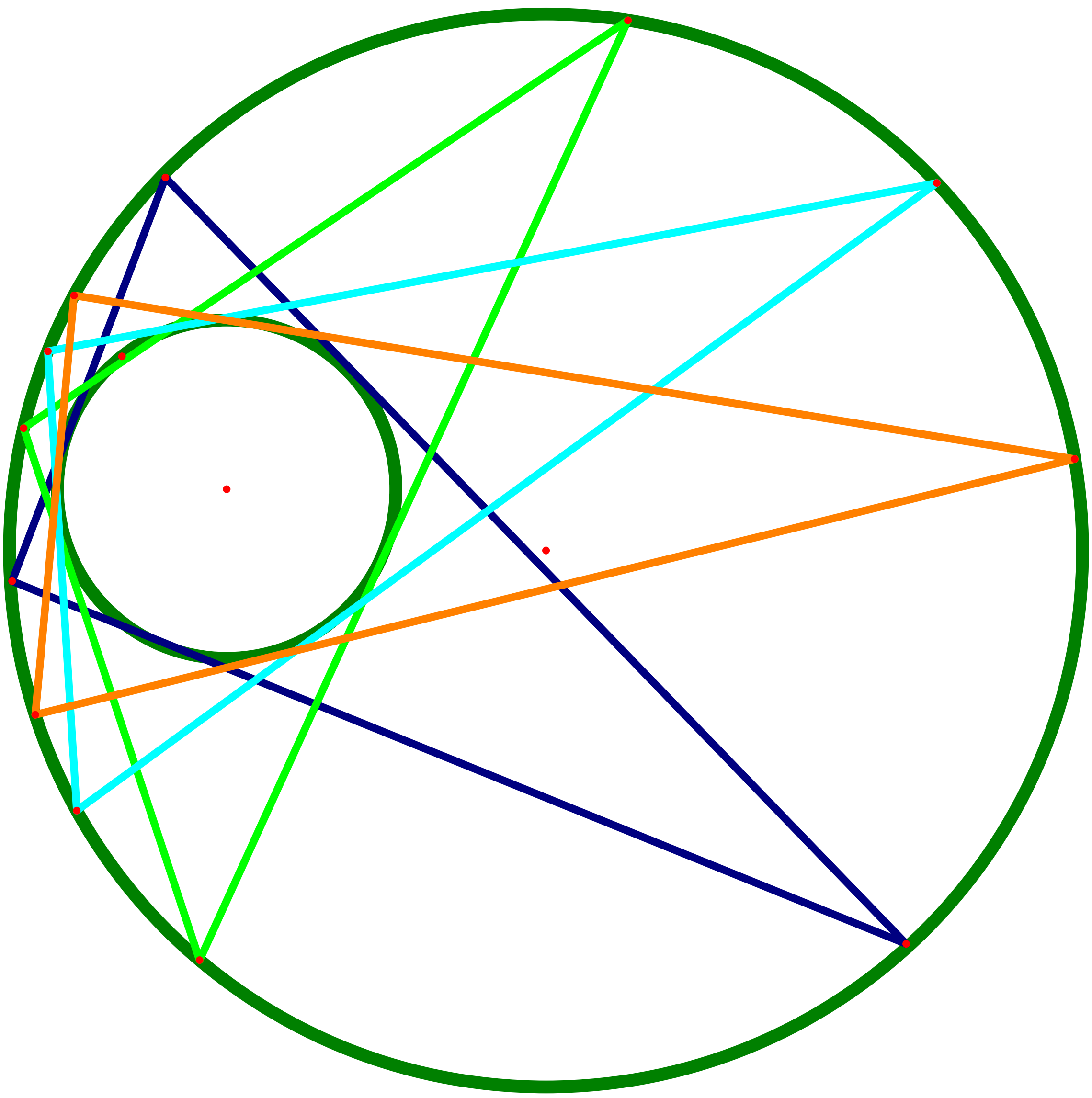
Acetate **A** with two circles (black)	Draw one triangle (red).

Class experiment

Paper **A** with 2 circles, Coloured pencils, x 15	*You may ask: How did he know where to start the triangle?*

When Napoleon retreated from Moscow after his disastrous Russian campaign at the beginning of the nineteenth century, a young lieutenant, left for dead on the battlefield, was found and, over a period of two years, nursed back to health. As he lay on his sickbed he began to think about the sort of geometry we have been discussing. What he realised was that it doesn't matter where you start because every one of the infinite number of possible points results in a triangle which closes up. The man was Jean-Victor Poncelet.

Try a few, using different colours so that they don't get mixed up.



But, again, that's not all.

E27 *Teacher demonstration*

Set OHP at an angle to the screen.
As with Pascasl's theorem, Poncelet's closure theorem works for any conics.

That was triangles. What about quadrilaterals?
Again there is a formula which allows me to fit a quadrilateral between two circles.

E28 *Teacher demonstration*

Acetate **B** with 2 circles (black)

Draw one quadrilateral (red).

Class experiment

Paper **B** with two circles, Coloured pencils, x 15

Again, try a few.

As you now expect, when I swivel the projector, the construction will still work, even though my circles are now ellipses.

E29 *Teacher demonstration*

Set the projector at an angle to the screen.

In fact Poncelet's closure theorem applies to any polygon between any pair of conic sections.

Paper **C** with two ellipses, Coloured pencils, x 15

E30 *Class experiment*

Try pentagons here.

In the special case of the triangle, the 'closure' theorem can be proved by the geometry of the circle and the incircle and circumcircle of the triangle. Poncelet's achievement was to generalise this result.

When we see a convincing three-dimensional scene in a cartoon or a computer game, we need to remember that, at the back of it, is projective geometry, cleverly implemented as computer code into the software used by the artist.

Paul Stephenson
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