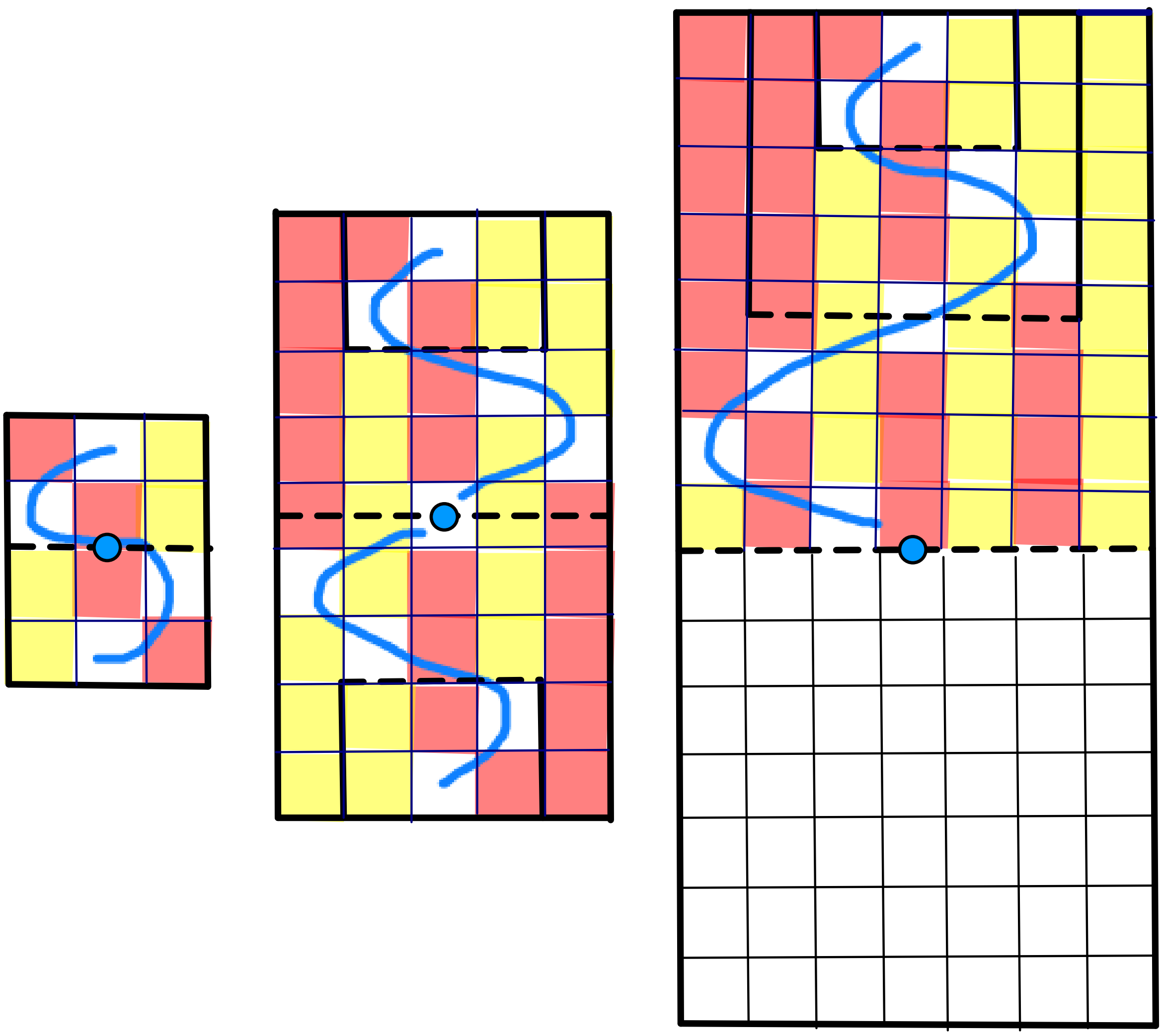
Topic: **Sequences**

Station: **Leapfrog**

In these diagrams each row is a bird's eye view of the zebra crossing as each move is made. So, reading down the page, a whole rectangle is a film, frame by frame, of the frogs swapping places: on the left, 1 frog of each colour; in the middle, 2; on the right, 3.

As with the Hanoi tower, there is a nested structure. The set of moves for n frogs is embedded in the set of moves for $n + 1$ frogs. As you see from the figure, if the rectangle for n is broken in two, you can fit the pieces in the top and bottom of the rectangle for $n + 1$. Notice also the half-turn symmetry of the whole diagram and the way the space meanders across it - you may like to copy and complete the unfinished diagram on the right.



To work out the number of moves, N, for n frogs, there are two approaches:

1. Empirical and inductive

We do the activity and complete a difference table. Below is the start of it. The three columns before the '0' suggest that the formula which fits has the form $N = an^2 + bn + c$. Below the difference table is a matching one with the correct n value substituted in our formula for each case.

Frogs each side, n	Moves	Difference	Difference	Difference
1	3			
		5		
2	8		2	
		7		0
3	15		2	

$a + b + c$		
	$3a + b$	
$4a + 2b + c$		$2a$
	$5a + b$	
$9a + 3b + c$		

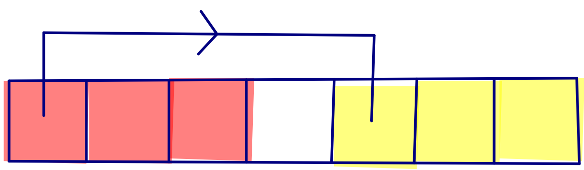
Our job is to find the coefficients a, b, c.

Substituting from right to left, we find $a = 1$, $b = 2$, $c = 0$, so our formula is $N = n^2 + 2n$.

2. Theoretical and deductive

This approach is more satisfactory because it not only gives us the formula but explains why it has the form it does.

Since a frog cannot jump another of the same colour, the frogs must maintain their order. This means each frog moves the same number of places, namely $n + 1$, as shown below:



Therefore the total number of places moved is the total number of frogs times the places moved by each, $2n \times (n + 1) = 2n^2 + 2n$.

For a red frog to reach its final place, it must pass n yellow frogs. Likewise the corresponding yellow frog must pass n red frogs. Two passes are achieved whenever a red frog jumps a yellow or a yellow frog jumps a red. Therefore the pair of frogs need to make n jumps between them. This accounts for a total number of n pairs $\times$ n jumps $= \boldsymbol{n^2}$ **jumps**. A jump accounts for a move of two places. So the total number of places accounted for by jumps is $2n^2$.

This means the number of places accounted for by slides = $(2n^2 + 2n) - 2n^2 = \mathbf{2n}$. Since a slide accounts for a move of 1 place, this is also the number of slides.

The total number of moves is therefore the number of jumps + the number of slides = $\mathbf{n^2 + 2n}$.