Interdissectible tilings

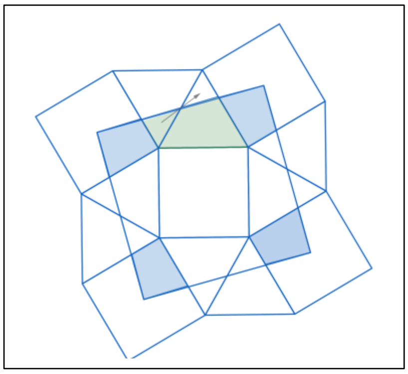
A popular activity in recreational mathematics is to cut up a regular polygon and reassemble the parts as another. The challenge is to do so using the fewest pieces.

In a semiregular tiling, an aggregate of regular polygons produces the whole tiling by translation. When a large part of the tiling is complete, this aggregate may be invisible. But the symmetry of the tiling tells us how we can join 4 points to make a parallelogram of the same area. This unit, what crystallographers call a primitive cell, will generate the whole tiling by translation in the same way the aggregate would.

We can dissect the aggregate and reassemble the parts to make this unit. In some cases there are few parts and we need few cuts to separate them. In this article we shall seek the repeat units which contain the fewest parts and need the fewest cuts regardless of whether they are primitive cells or even parallelograms. When this unit differs from the primitive cell, we shall have a tiling which is different from the original because the parts are jumbled up. We shall call the original tiling and the reordered tiling 'interdissectible', simply meaning that we can dissect one tiling into the other.

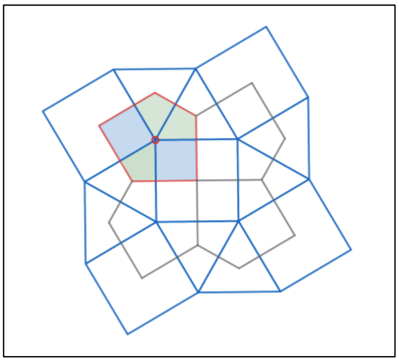
First we need to spot the 'invisible' aggregate. There are several equivalent methods. We shall apply them to a single example, 3.3.4.3.4 (or $3^2.4.3.4$). In a semiregular tiling the vertices are identical. The notation tells us that, making a tour of a vertex (anticlockwise, say), we meet in order, two triangles, a square, another triangle and another square.

1.



Here is the primitive cell. It contains a complete square and one in 4 congruent quarters. If we move the small triangle as shown by the arrow, we complete a triangle on each side of the central square. The ratio of triangles to squares is therefore 2 : 1.

2.

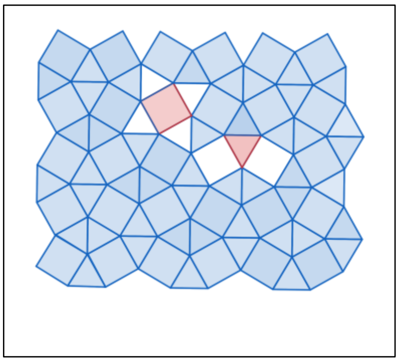


Joining the polygon midpoints, we produce a dual tiling. In the case of the semiregular tilings, this is called a Laves tiling. All the tiles like the red one are congruent. The fractions of the polygons it contains therefore represent the fractions in the tiling as a whole. We have 3 thirds of a triangle to 2 quarters of a square, scaling up to the ratio 2 : 1.

The red tile can also be described as a Voronoi polygon, the region containing points as near as, or nearer to, a central point than they are to any other point in the plane.

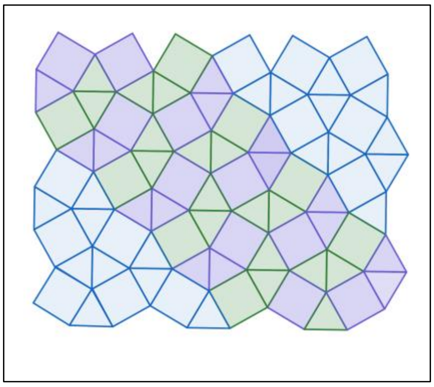
We can also use our original symbol, 3.3.4.3.4, swapping the integers for their reciprocals. The triangles account for 3 thirds, the squares for 2 quarters.

3.

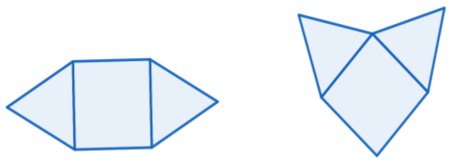


Each square is bordered by 4 triangles.
Each triangle is bordered by 2 squares.
So, again, there are twice as many triangles as squares.

4.

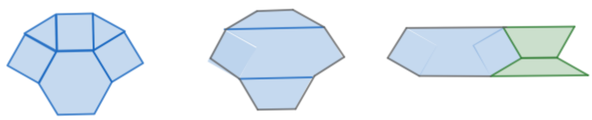
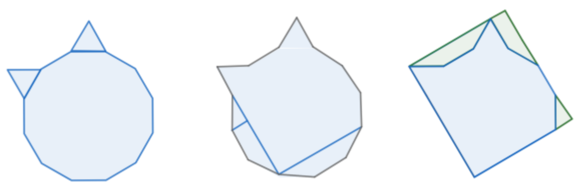


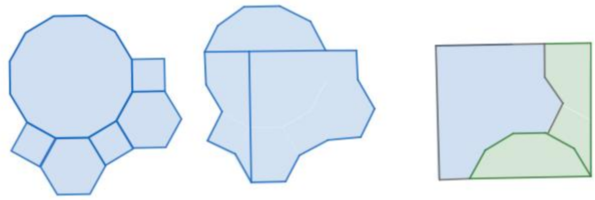
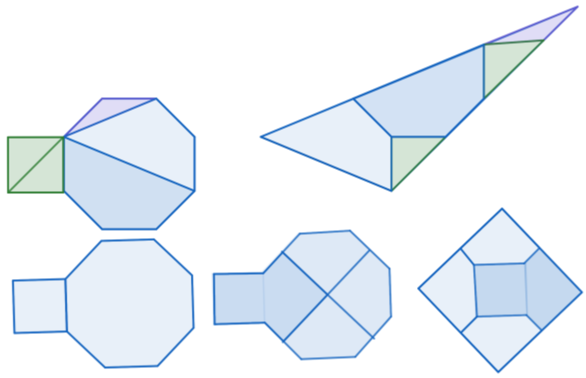
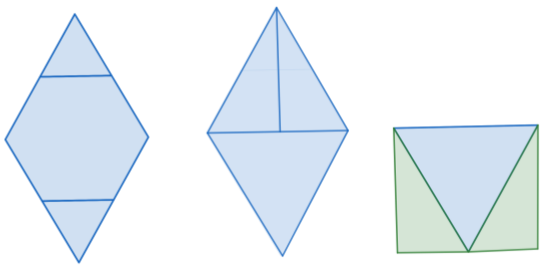
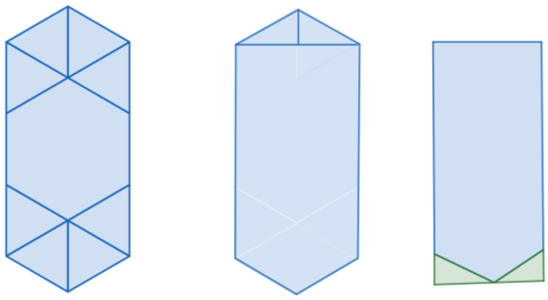
We spot that the 'elf's head' tessellates. The same is true of the 'boat' and the 'cat's head':

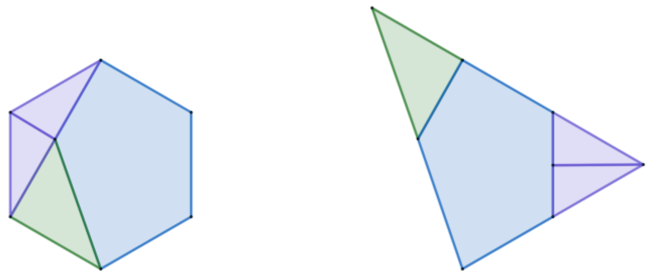
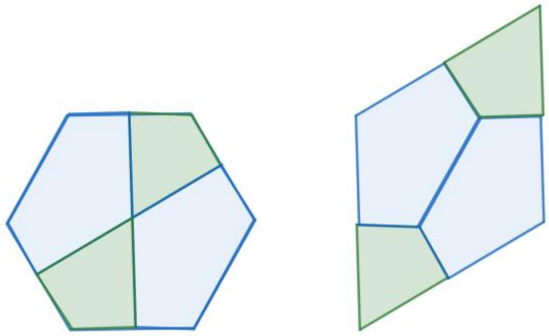
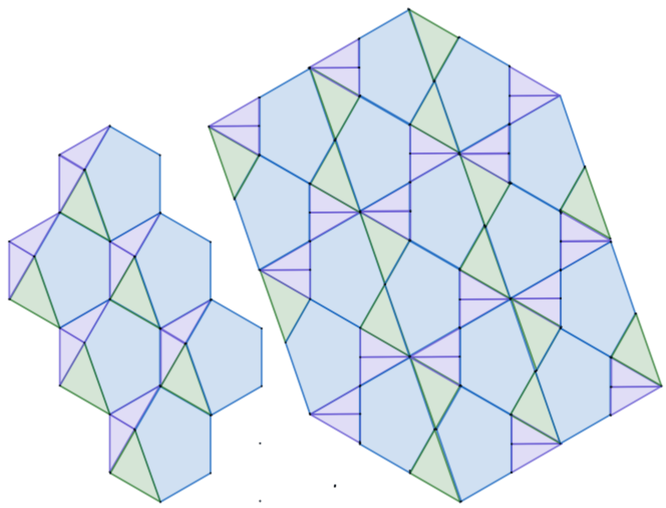


Try them!

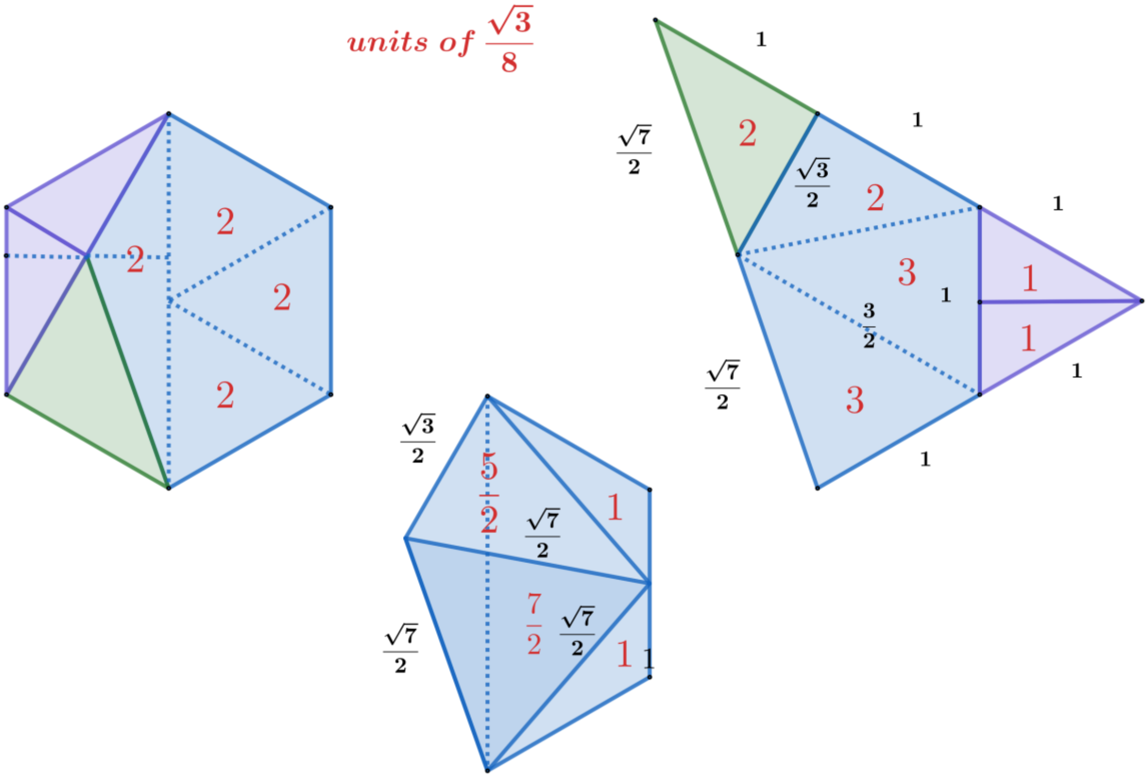
With the counting out of the way, we are ready to dissect. Regular tilings are uniform tilings where there is just one polygon. In all three such cases the primitive cell is also the most economical. But we have added to the table 6^3 because the alternative is interesting.

Tiling	Ratio	Dissection	Cuts	Pieces
3.4.6.4	3s : 4s : 6s 2 3 1		2	3
3.12^2	3s : 12s 2 1		3	4

4.6.12	4s : 6s : 12s 3 2 1		2	3
4.8^2	4s : 8s 1 1		4 2	5 4
$3^2.4.3.4$ (This also serves for $3^3.4^2$)	3s : 4s 2 1		1	2
3.6.3.6	3s : 6s 2 1		2	3
$3^4.6$	3s : 6s 8 1		2	3

6^3		  Here is part of the tiling that results from the upper dissection: 	3 2	4 4

Here are some relations within the triangle produced from 6^3:



Paul Stephenson
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