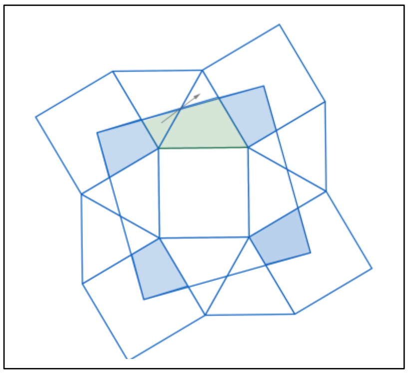
Interdissectible tilings

A popular activity in recreational mathematics is to cut up a regular polygon and reassemble the parts as another. The challenge is to do so using the fewest pieces. We shall do so for complete tilings. To do this we need to identify the unit which repeats to form the tiling.

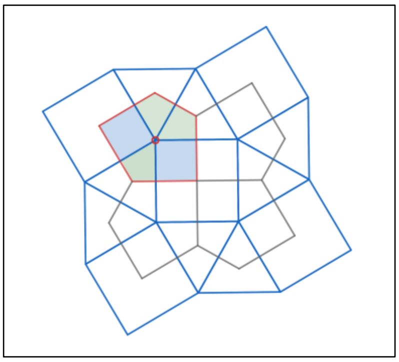
There are several equivalent methods. We shall apply them to a single example, 3.3.4.3.4 (or $3^2.4.3.4$). In a semiregular tiling the vertices are identical. The notation tells us that, making a tour of a vertex (anticlockwise, say), we meet in order, two triangles, a square, another triangle and another square.

1.



Here is the primitive cell. It contains one square which is complete square and one in 4 congruent quarters. If we move the small triangle as shown by the arrow, we complete a triangle on each side of the central square. The ratio of triangles to squares is therefore 2 : 1.

2.

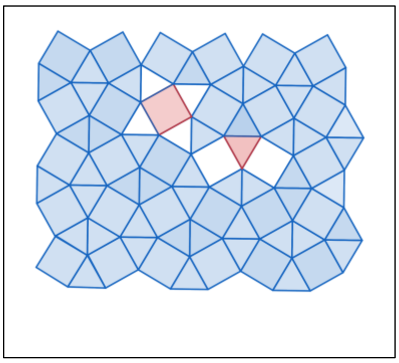


Joining the polygon midpoints, we produce a dual tiling. In the case of the semiregular tilings, this is called a Laves tiling. All the tiles like the red one are congruent. The fractions of the polygons it contains therefore represent the fractions in the tiling as a whole. We have 3 thirds of a triangle to 2 quarters of a square, scaling up to the ratio 2 : 1 again.

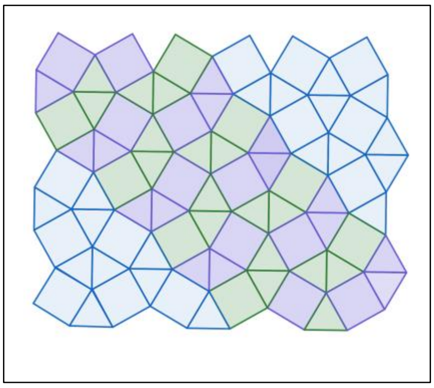
The red tile can also be described as a Voronoi polygon, the region containing points as near as, or nearer to, a central point than they are to any other point in the plane.

3. We can also use our original symbol, 3.3.4.3.4, swapping the integers for their reciprocals. The triangles account for 3 thirds, the squares for 2 quarters.

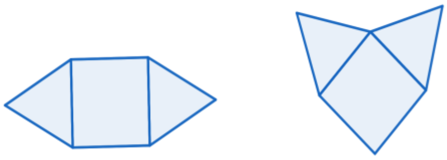
4.



Each square is bordered by 4 triangles.
Each triangle is bordered by 2 squares.
So, again, there are twice as many triangles as squares.

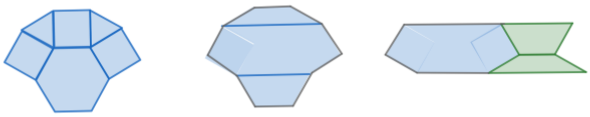
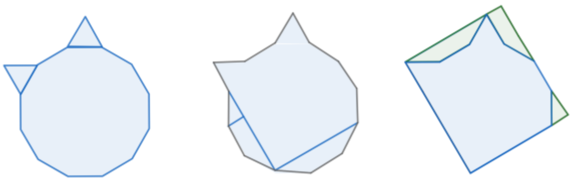
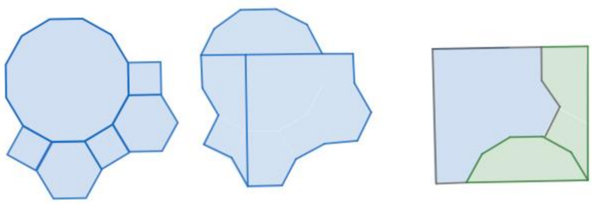
5. 

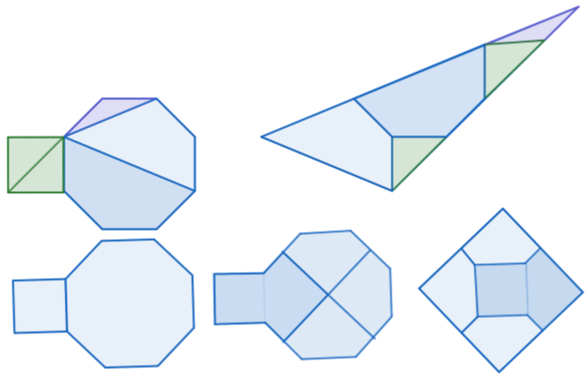
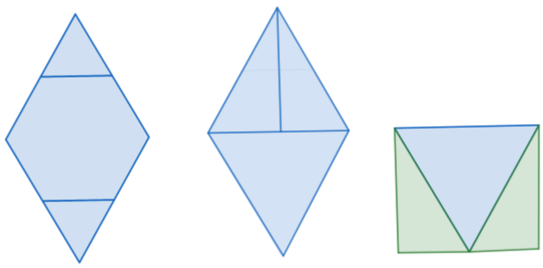
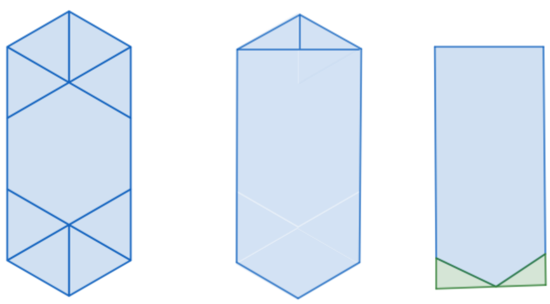
We spot that the ‘elf’s head’ tessellates. The same is true of the ‘boat’ and the ‘cat’s head’:

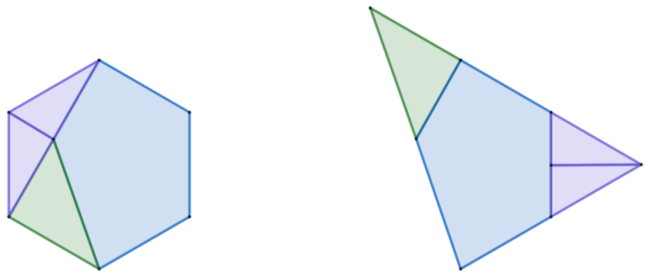
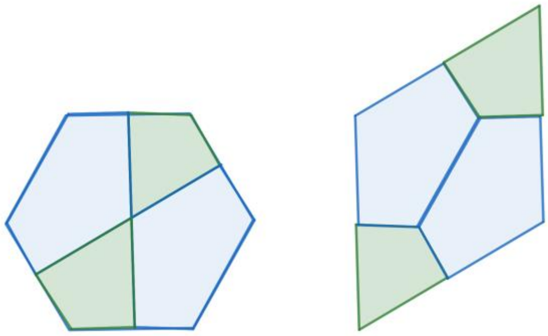
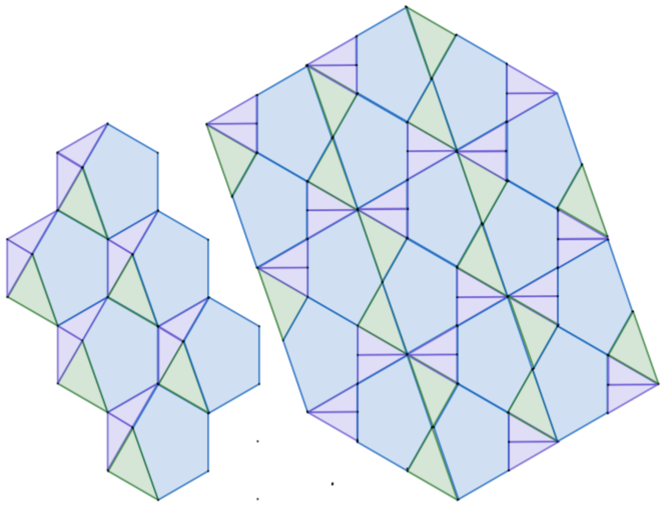


Try them!

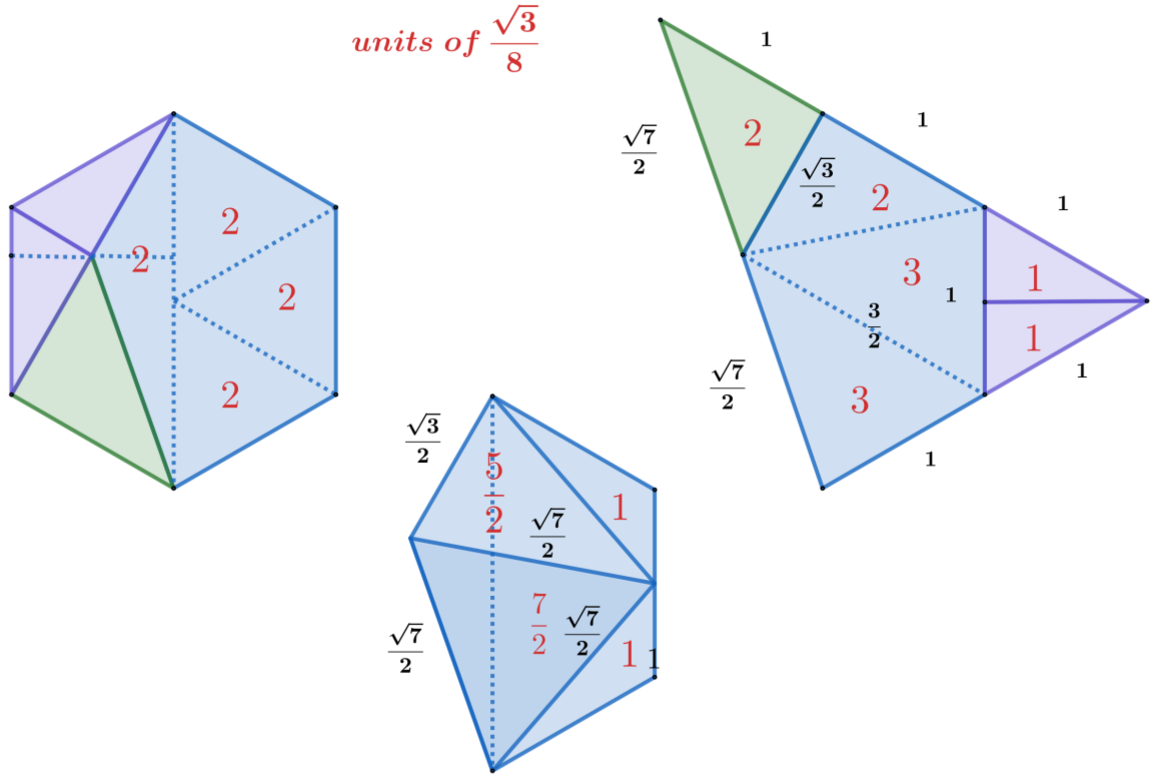
With the counting out of the way, we are ready to dissect. Regular tilings are uniform tilings where there is just one polygon. In all three such cases the primitive cell is also the most economical. But we have added to the table 6^3 because the alternative is interesting.

Tiling	Ratio	Dissection	Cuts	Pieces
3.4.6.4	3s : 4s : 6s 2 3 1		2	3
3.12^2	3s : 12s 2 1		3	4
4.6.12	4s : 6s : 12s 3 2 1		2	3

4.8^2	4s : 8s 1 1		4 2	5 4
3^2.4.3.4 (This also serves for $3^3.4^2$)	3s : 4s 2 1		1	2
3.6.3.6	3s : 6s 2 1		2	3
$3^4.6$	3s : 6s 8 1		2	3

6^3		  Here is part of the tiling that results from the upper dissection: 	3 2	4 4

Here are some relations within the triangle produced from 6^3:



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13.6.23