‘Infill’ tilings

Introduction

In (Stephens, Stephens, Parramore, 2022), the authors find all the ways regular polygons fit round a vertex. This amounts to finding all the solutions to this equation:

$$\frac{m-2}{2} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots + \frac{1}{n_i} + \cdots + \frac{1}{n_m},$$

where $3 \leq m \leq 6$, and n_i is the number of sides of each polygon.

Here is their table. I’ve coloured blue the vertices found in Archimedean tilings, and red, those lonely cases looking for a tiling to belong to.

$m = 3$	(3,7,42)	(3,8,24)	(3,9,18)	(3,10,15)	(3,12,12)
	(4,5,20)	(4,6,12)	(4,8,8)		
	(5,5,10)				
	(6,6,6)				
$m = 4$	(3,3,4,12)	(3,3,6,6)	(3,4,4,6)	(4,4,4,4)	
$m = 5$	(3,3,3,3,6)	(3,3,3,4,4)			
$m = 6$	(3,3,3,3,3,3)				

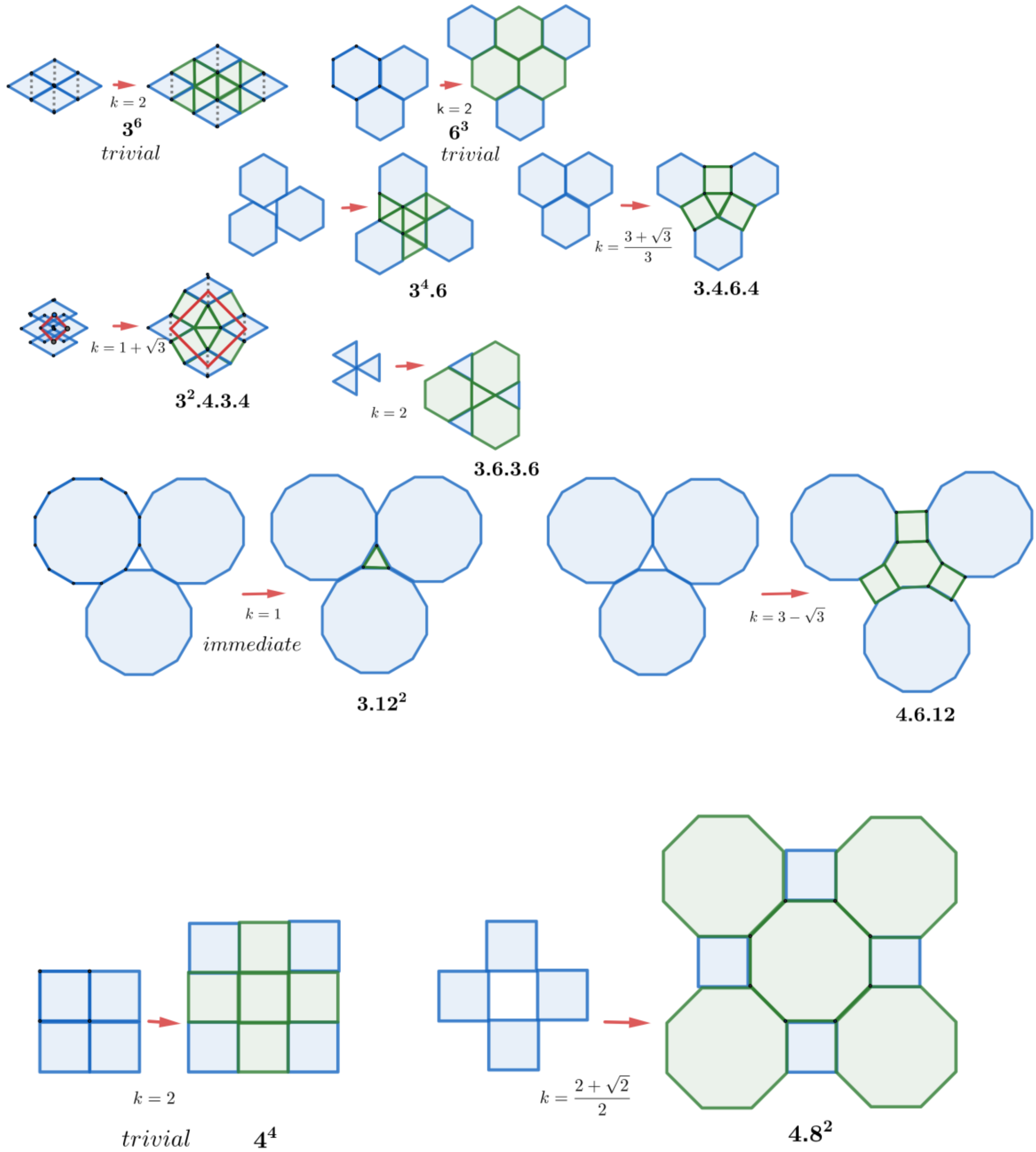
In trying to accommodate red vertices, which we’ll call ‘non-Archimedean’, I have been led to a construction method which generates almost all the Archimedean tilings and those which can include red vertices with the least possible reduction in symmetry (though I shall not attempt to prove that my solutions are the best in that sense). All the tilings we discuss are edge-to-edge with the exception of one: that required to generate $3^4.6$. All edge lengths are equal except where stated. The Archimedean tilings have a stronger property. They are *isotoxal*: every edge can be mapped onto another edge by a symmetry of the tiling. They are also *isogonal*: every vertex can be mapped onto another vertex by a symmetry of the tiling. This last property makes the Archimedean tilings *uniform.* If there are *k* different vertex types but a vertex of one kind can be mapped onto one of the same kind by tiling symmetries, the tiling is *k-uniform.* If, in an Archimedean tiling, all the regular polygons are the same, we have a *regular* Archimedean tiling; where there is more than one variety, we have a *semiregular* Archimedean tiling. In the standard definition, all the tiles of a *k*-uniform tiling are regular polygons. We shall need to extend this definition to include irregular forms.

The expand-&-fill construction

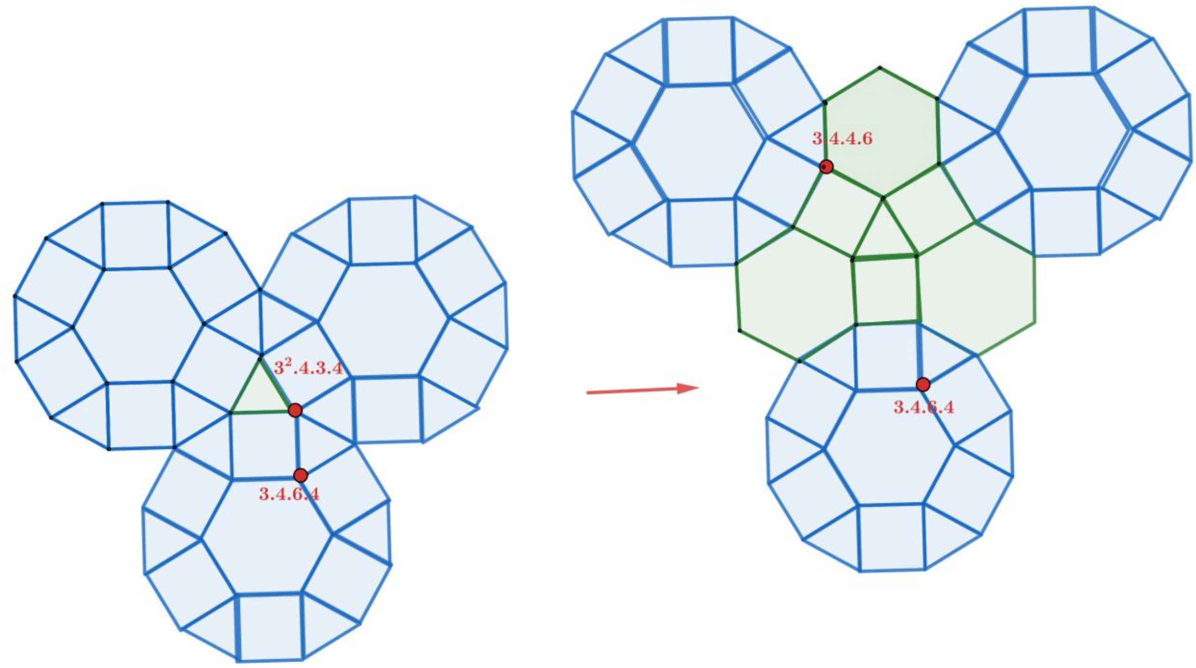
We start with an infinite periodic arrangement (a packing or tiling or covering) derived from a particular arrangement of regular polygonal tiles pinned by their centres to a rubber sheet which we stretch so that it scales uniformly with factor *k*, the tiles maintaining their orientation. (That is to say, any figure drawn on the sheet is enlarged. Think of little galaxies in an expanding universe). We have so chosen the initial arrangement that, at a particular *k* value, we can fill with more regular polygonal tiles the gaps which open up. We shall call this method the ‘expand-&-fill’ construction, the original tile pattern, the ‘seed’, the polygons we insert, the ‘fill’, and the result an ‘infill tiling’. The *crystallographic restriction* means our seed forms can only have rotation axes of orders 2, 3, 4, 6.

Generating the regular infill tilings

We shall first use our construction method to generate (most of) the Archimedean tilings. Notice that the patterns which start with part of a regular tiling merely repeat it. The tiling we can’t produce is $3^3.4^2$. This one is unique in having only the symmetries possible for a frieze. It cannot be the result of a uniform scaling, which requires equal scaling in two perpendicular directions. In the case of $3^4.6$ we have to rotate the rubber sheet: the transformation of the sheet is not a similarity but a spiral similarity.



We shall call the above ‘*regular* infill tilings’, ‘regular’ because the fill consists only of regular polygons. In some cases we can iterate the expand-&-fill process. With an Archimedean patch as seed, we can produce k-uniform tilings for $k > 1$. For example, starting with a patch of 3.4.6.4 corresponding to a regular 12-gon, we can immediately produce $3^2.4.3.4$; 3.4.6.4 and, by a further expansion, the 2-uniform tiling $3^2.4.3.4$; $3.4^2.6$:



Note that swapping the ‘4’ and the ‘6’ in 3.4.6.4 produces a non-Archimedean vertex (written 3.4.4.6 to emphasise the swap).

We shall now construct *irregular* infill tilings which can accommodate our six non-Archimedean vertices. These necessarily include irregular polygons. But we shall first locate them in the bigger picture.

Where irregular infill tilings fit in a tiling classification

In terms of the standard nomenclature, to achieve our irregular infill tilings, we must relax conditions and descend from level A, *regular* tilings, through B, *semiregular* tilings, C, *k-uniform* tilings, D, our *infill* tilings, down to E, tilings we shall derive from them where, though one of the polygons is irregular, ignoring sense, there is only one vertex type. We shall extend the term *uniform* to cover tilings where, though each of the *k* types of vertex can be mapped onto another of the same kind by symmetries of the tiling, not all the polygons are regular.

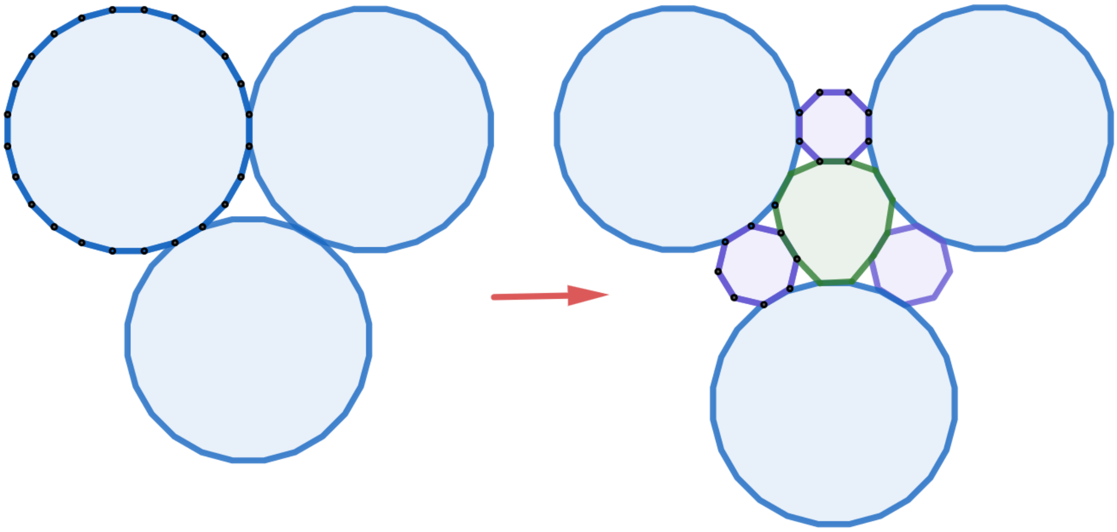
Level	All vertices the same (except for sense)?	All polygons regular?	Polygons of just one kind?	Reflective and rotational symmetries?
A	Yes	Yes	Yes	Yes
B	Yes	Yes	No	Yes
C	No	Yes	No	Yes
D	No	No	No	Yes

E	Yes	No	No	Yes

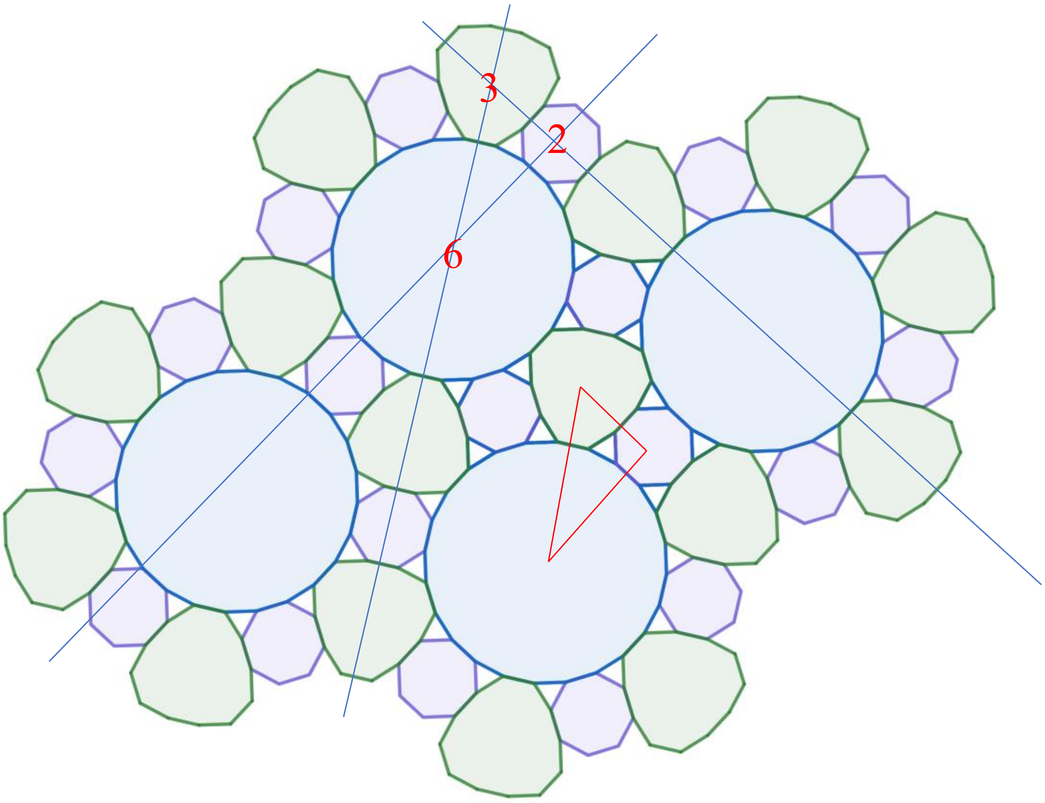
Generating the irregular infill tilings

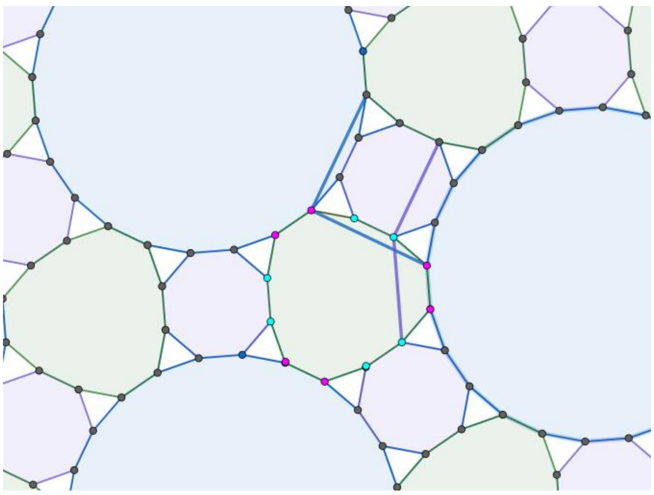
We now generate each of the 6 irregular infill tilings and discuss their symmetries.

(1) Here is the expand-&-fill diagram for the tiling which includes the vertex (3,8,24). In fact **all** the vertices in this tiling are (3,8,24) in one sense or the reverse. As we go round an octagon or a 24-gon, the sense alternates. As we go round the green shape, we have two clockwise vertices alternating with two anticlockwise ones.



The infill tiling has the full symmetry of 6^3. Here is a larger patch. The symmetry elements are marked. It has Conway notation *632.

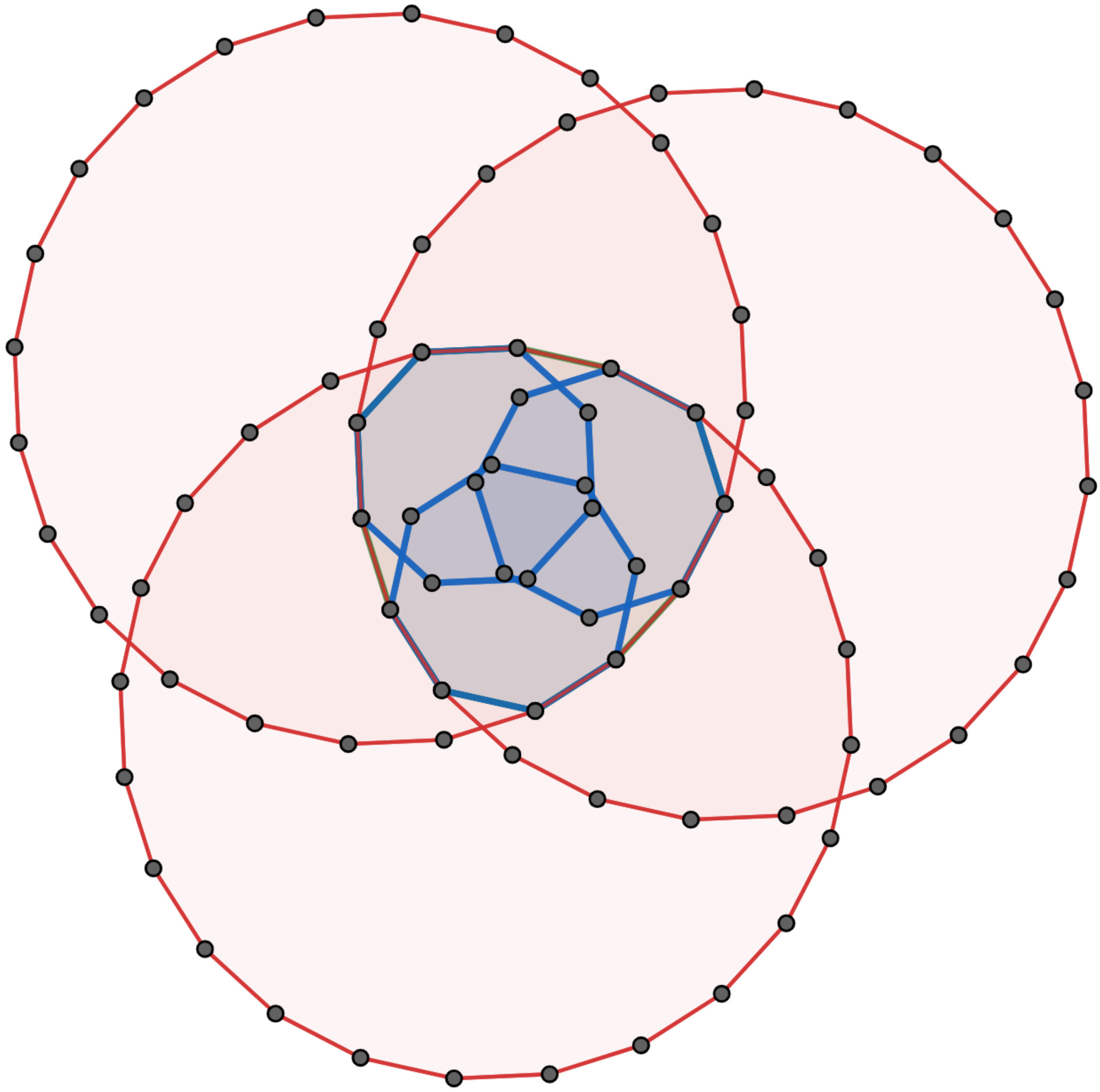




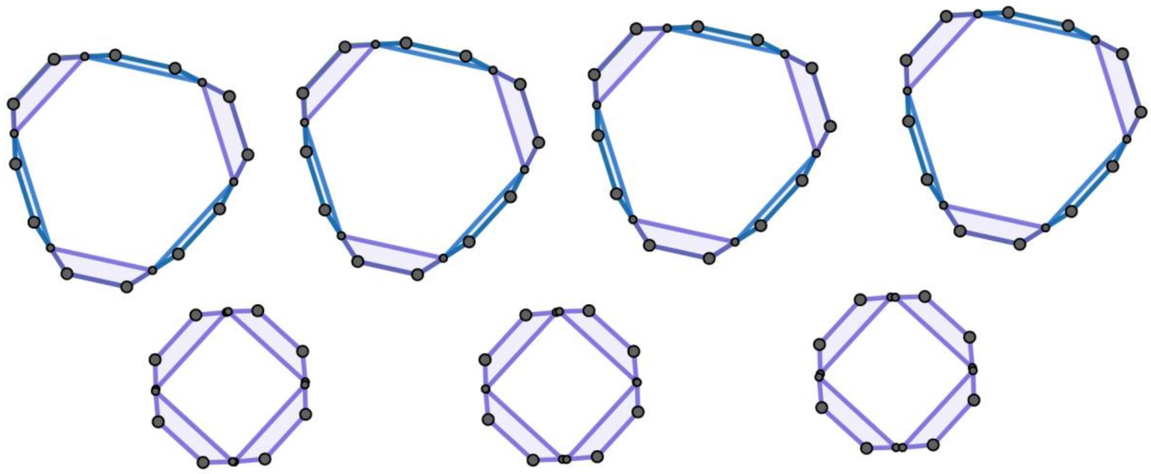
The red triangle is part of the dual tiling.

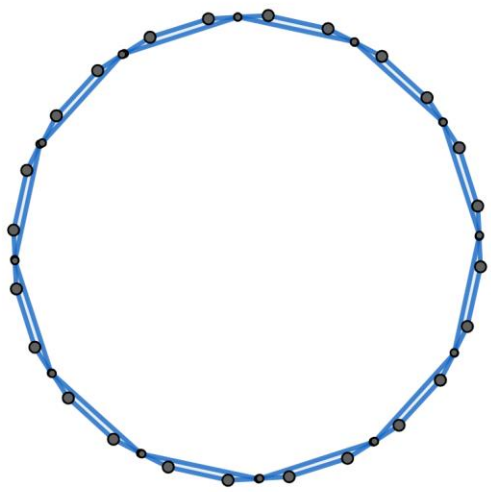
Here we zoom in on the green shape. It is a hybrid of the octagon and 24-gon: the 'blue' segment of the green shape is shared with the 24-gon; the 'lilac' segment with the octagon. Equal angles in the green shape are colour-coded. It is a dodecagon with the symmetries of the equilateral triangle.

This is what the green shape looks like if we draw in complete octagons and 24-gons. If side k belongs only to an octagon, side $(k+2)$ belongs only to a 24-gon and side $(k+1)$ is common to the two. We shall call the green shape a *3*-form, to be defined later.

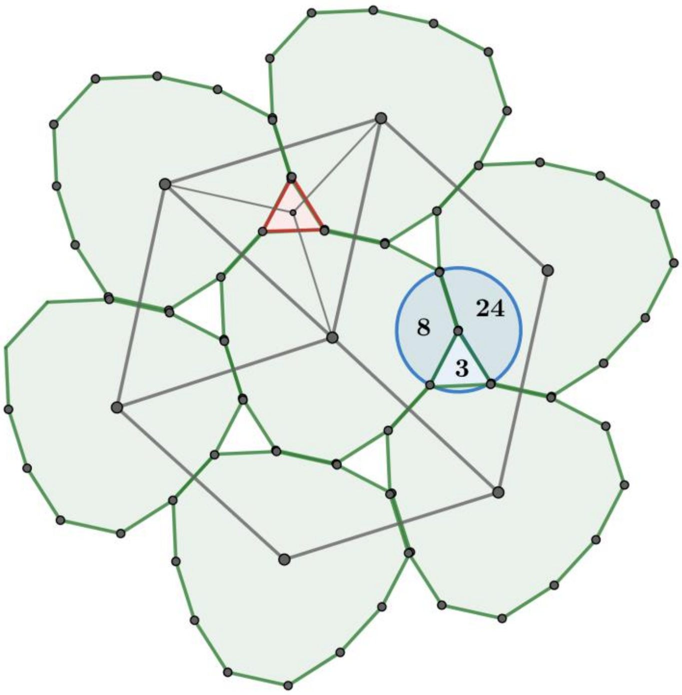


In the next figure we show how parts of four of the *3*-forms can be dissected to produce parts of three octagons and one 24-gon. At the 3-form centres we have equiangular hexagons with sides of alternate lengths; at the octagon centres we have squares; at the 24-gon centre, a regular dodecagon.





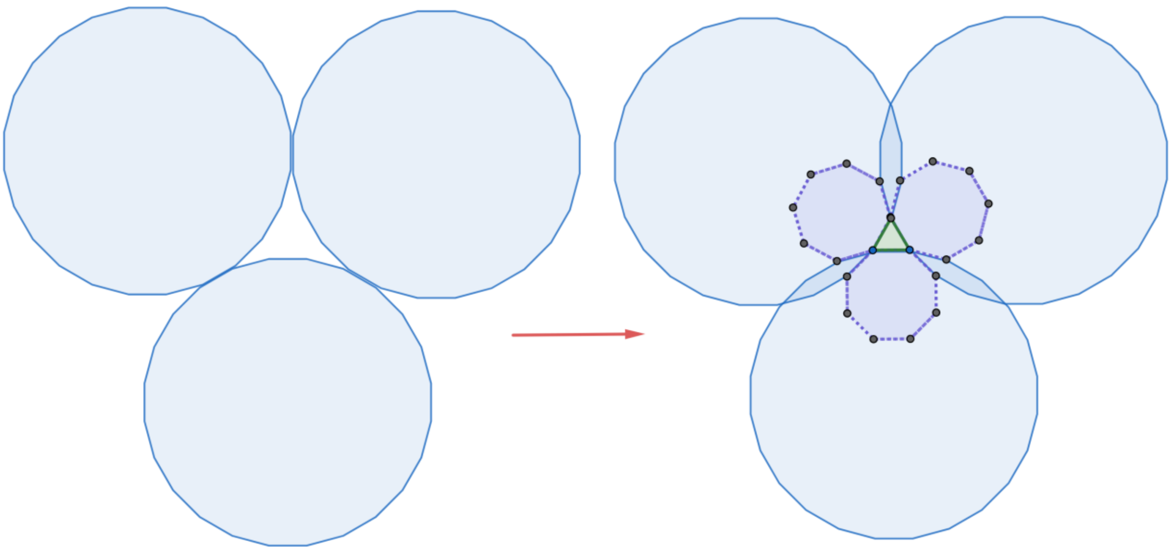
The *3*-form has accommodated itself to the equilateral triangle, regular octagon and regular 24-gon: at a cyan vertex we have a triangle and an octagon - the third angle must therefore be the interior angle of the third shape, the 24-gon; at a magenta vertex we have a triangle and a 24-gon - the third angle must therefore be the interior angle of the third shape, the octagon. The chameleon-like nature of the *3*-form means that we can tile the plane just with *k*-forms and equilateral triangles. This is our level E tiling.



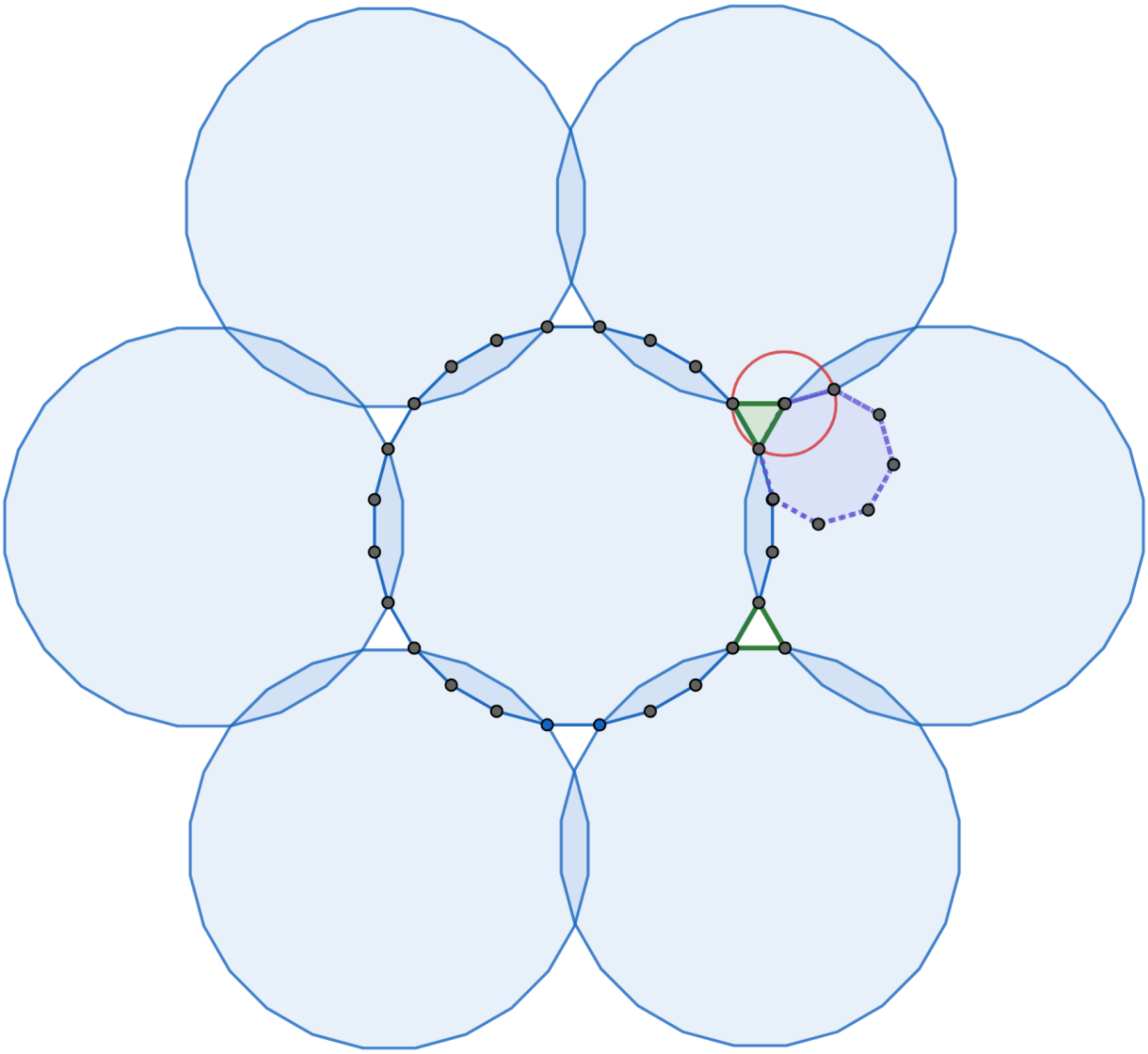
The result is the Archimedean tiling 3.12^2 deformed. The triangle centres are still centres of rotation symmetry of order 3, the dodecagon centres are still centres of rotation symmetry of order 3, but mirror lines run only through the dodecagon centres. It has Conway symbol 3*3.

Notice how the *3*-forms fit together so that each of the 3 angles is present at a vertex. If we accept the *3*-form as a valid shape, we have a uniform tiling. The vertices belonging to adjacent triangles have opposite *sense*.

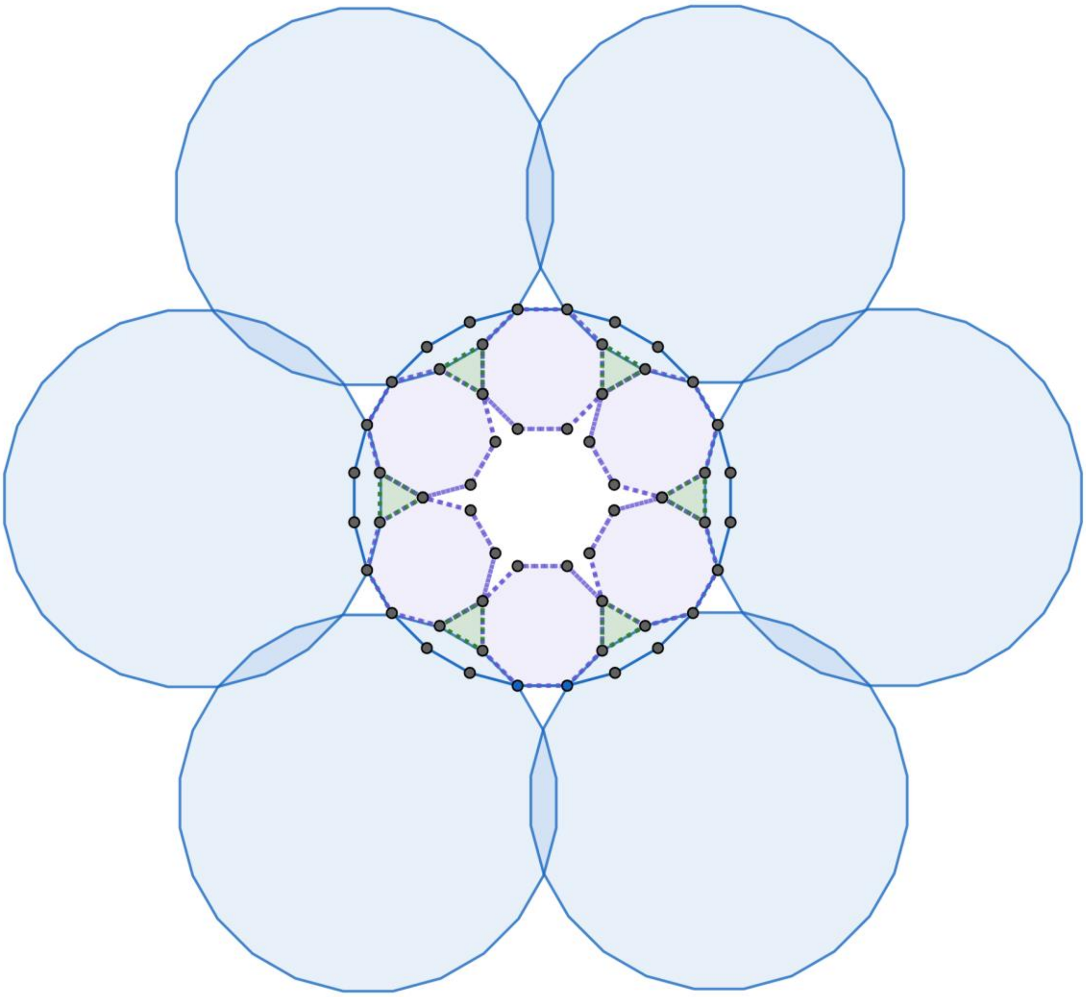
But we can accommodate non-Archimedean vertices in a different way. This is an alternative expand-&-fill, or rather contract-&-fill, diagram for (3,8,24).



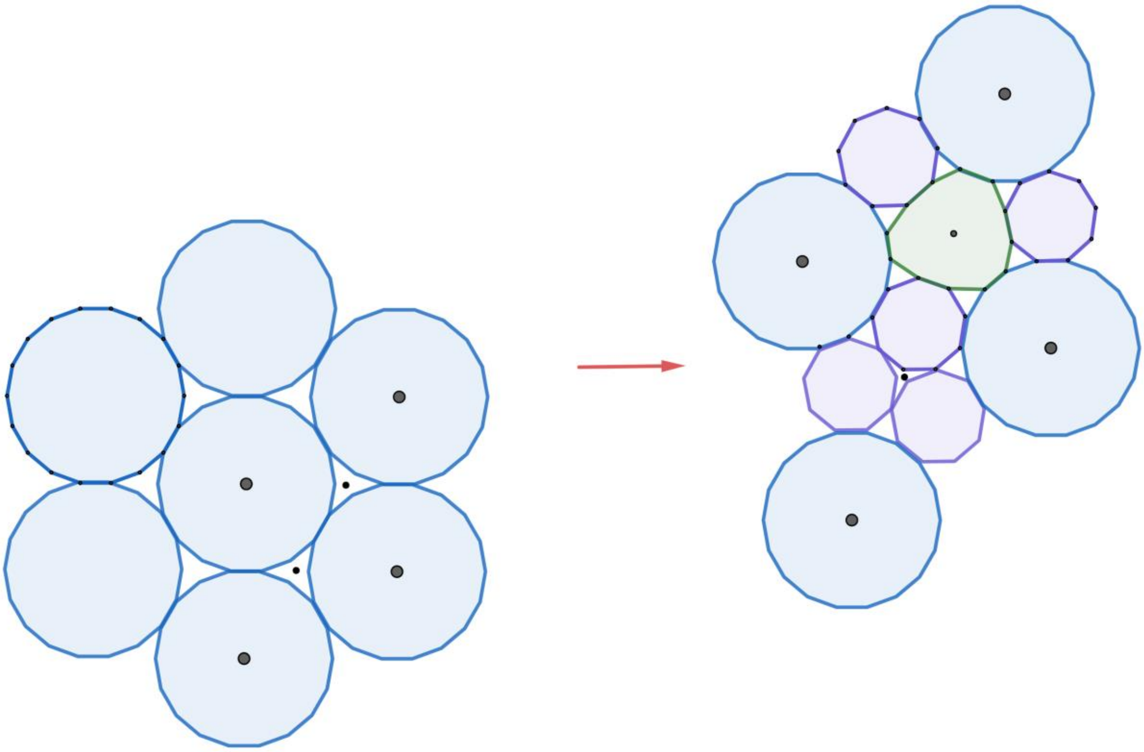
We pass from a packing to a partial covering. The next figure shows a vertex. We can notate the result 3.(8).(24), using brackets to show the polygons which overlap. If we decide to count only triangle vertices, and we call any patterning with tiles, whether packing, tiling or covering, a *tile pattern*, we have a uniform tile pattern. This remark applies also to the next two examples.



In fact 6 octagons pack inside the 24-gon so that they share vertices, leaving equilateral triangles between them:



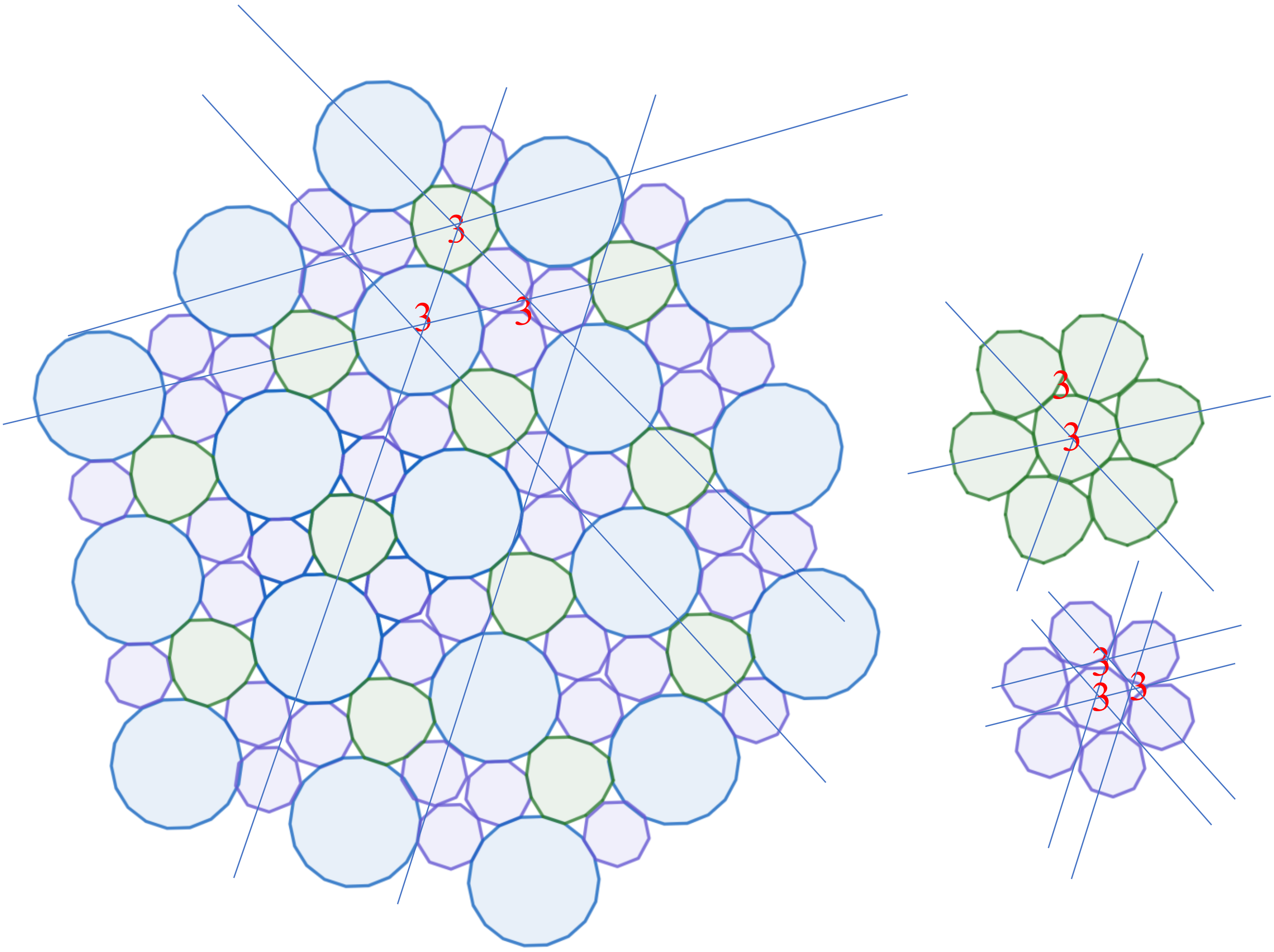
(2) Here is the expand-&-fill diagram for the tiling which includes the vertex (3,9,18). This time, at the centres of the triangles defined by 3 blue shapes in contact in the seed, we have 3-pointed stars alternating with *3*-forms:



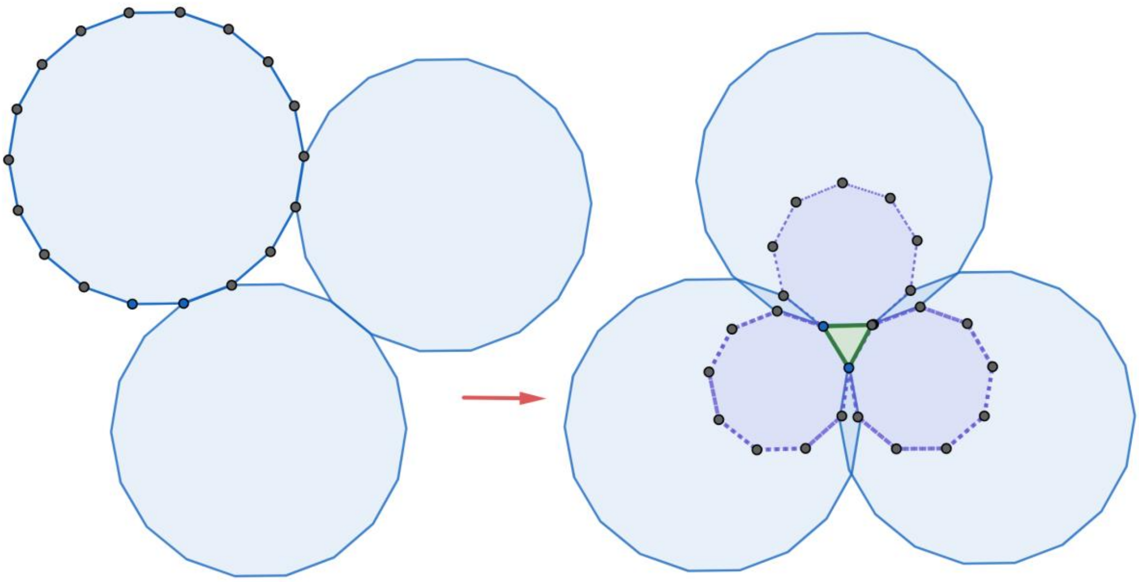
We have the same set of angles at each vertex except star points. This gives us two vertex types and the tiling is 2-uniform.

The symmetry elements are marked on the figures below. The main tiling has Conway symbol *333.

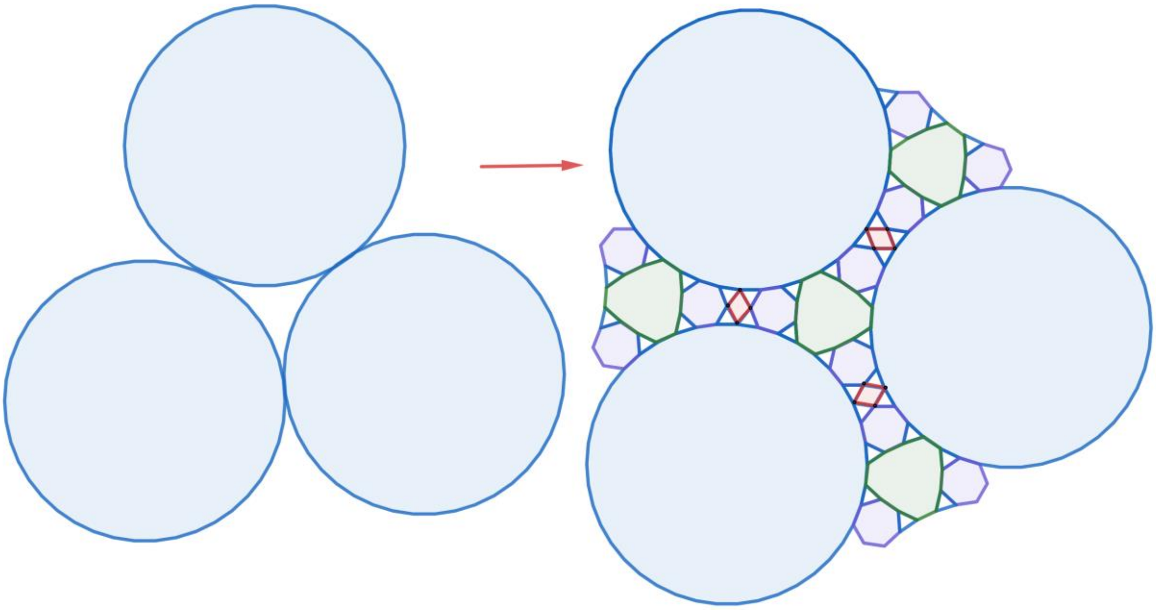
Although the *3*-form here has a shape which differs from the previous dodecagon, again we can use it to make a uniform tiling with the triangle. Again it is a distorted 3.12^2 and has Conway symbol 3*3. This time we can also make a tiling with stars, triangles and nonagons which is 2-uniform and has the same symmetries as the main tiling.



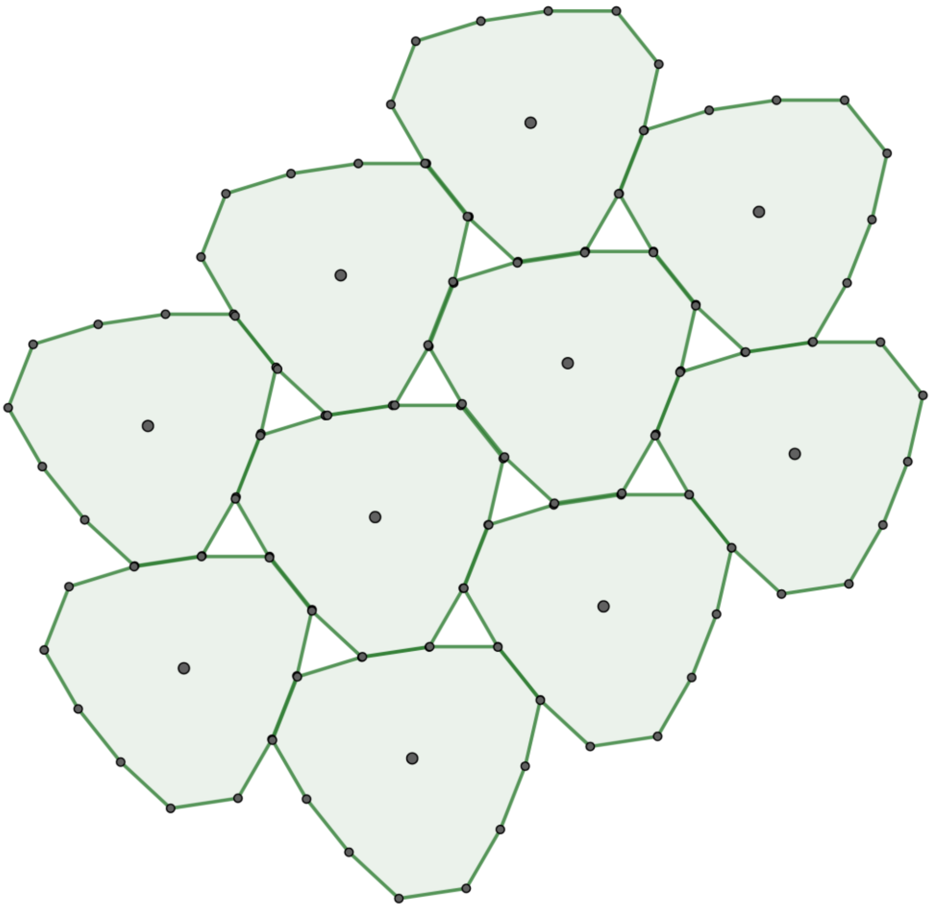
Here is the alternative expand-&-fill diagram for (3,9,18), producing the tiling 3.(9).(18):



(3) Here is the expand-&-fill diagram for the infill tiling which includes the vertex (3,7,42):

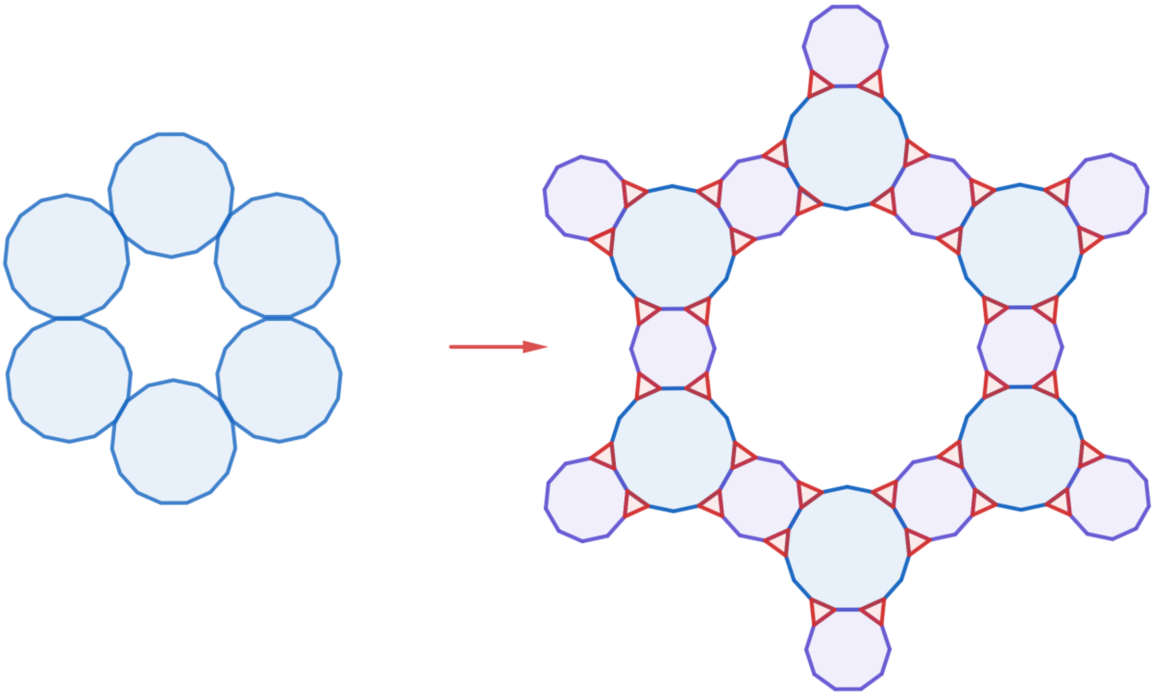


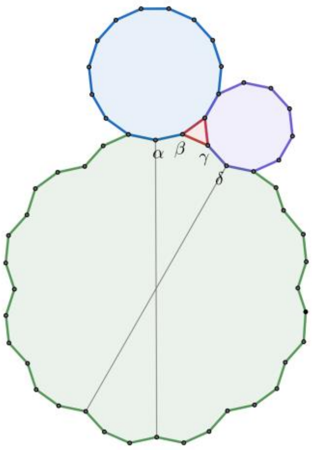
Again we have *3*-forms and one other irregular shape, this time a rhombus. This shape lies at the midpoints of lines joining blue centres, at the midpoints also therefore of the lines joining green centres. The overall symmetry is again that of 6^3, so has Conway symbol *632. There are vertices of 3 kinds: the original type and those including rhombus vertices. It is 3-uniform. Here is the derived tiling of triangles and *3*-forms, which has the same symmetries as the analogous figures already described.



The alternative construction is similar to those preceding and leads to 3.(7).(42).

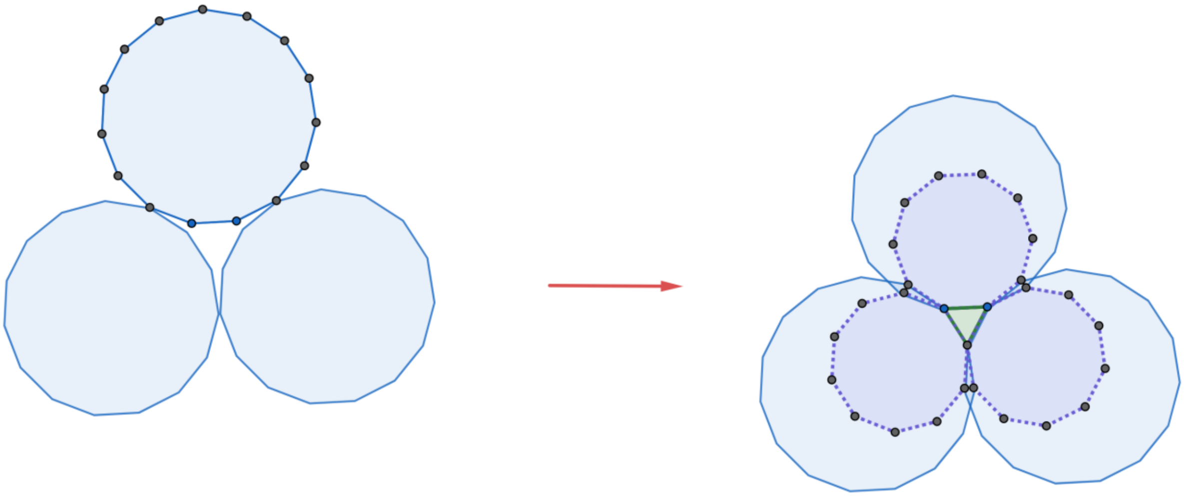
(4) Here is the expand-&-fill diagram for the tiling which includes the vertex (3,10,15):



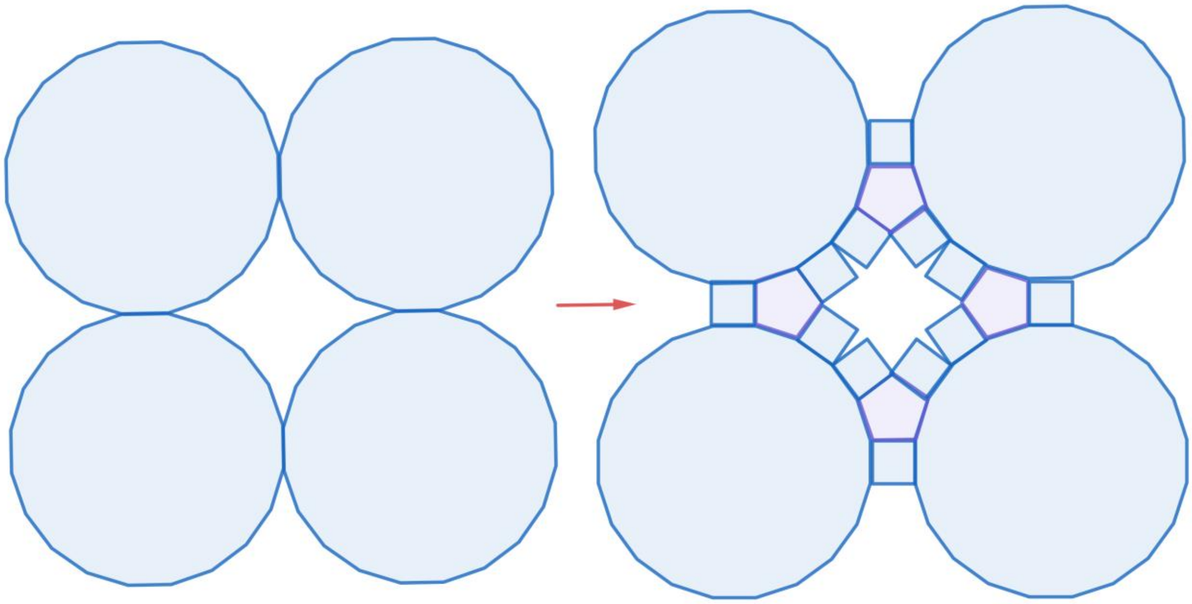


On the left is the central tile shown separately, with two of the 6 symmetry axes and the four vertex types additional to (3,10,15). It is a 36-gon. There are 6 vertices of type α, 6 of type δ, 12 of type β, 12 of type γ. We shall call it a *6*-form. When we come to notate the tilings, types α and δ will not count as vertices. The tiling is therefore 3-uniform. It has Conway symbol *632.

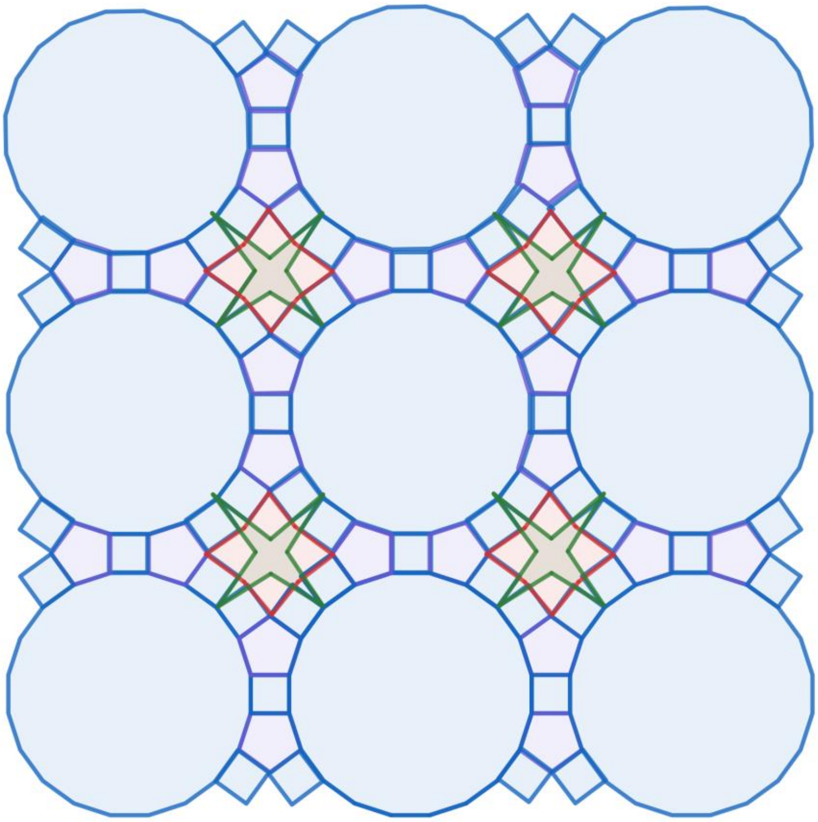
The alternative expand-&-fill construction in this case starts with a seed in which the polygons touch vertex-to-vertex and so as to enclose just one of the two possible Kepler stars.



(5) Here is the expand-&-fill diagram for the tiling which includes the vertex (4,5,20):



Here is more of the tiling to show how the central tile is made up:

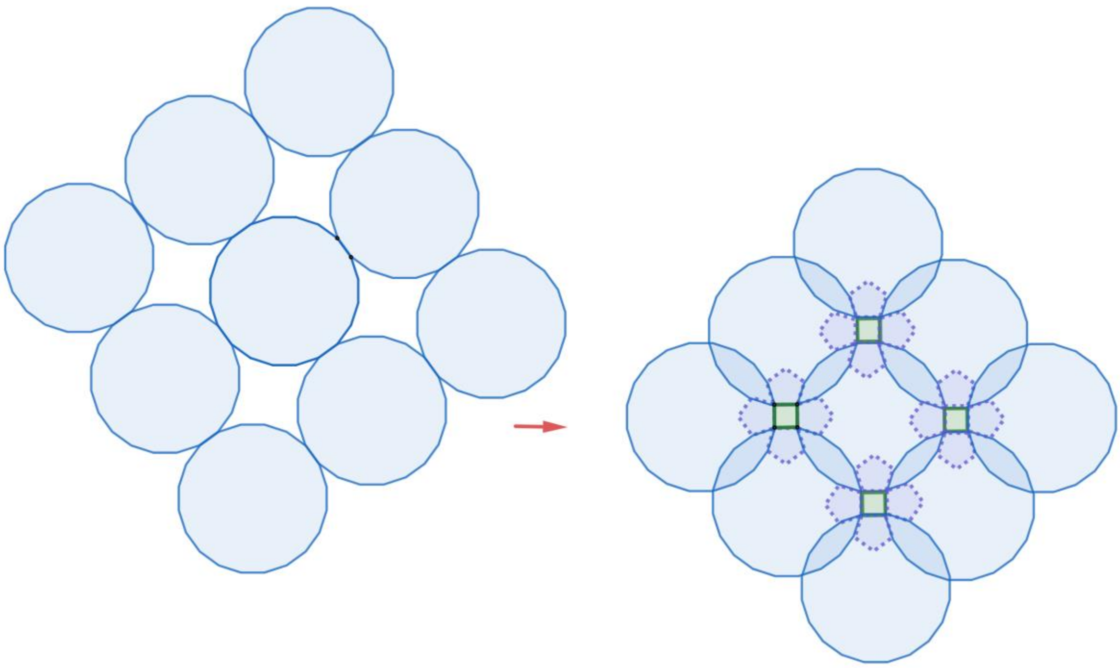


We shall call the tile a rhombic 2-star, to be defined later.

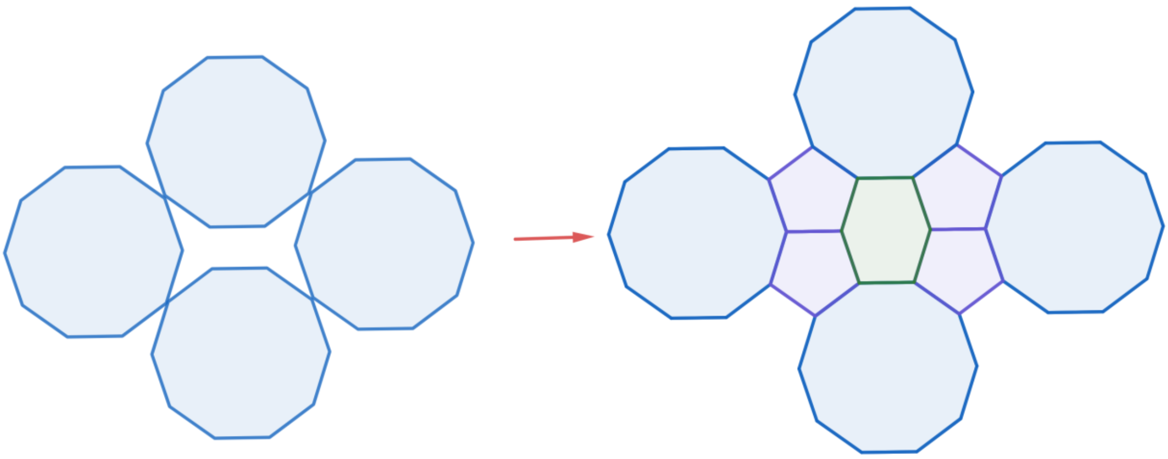
The tiling has the symmetry of 4^4, and Conway symbol *442.

There are 3 vertex types altogether. The tiling is 3-uniform.

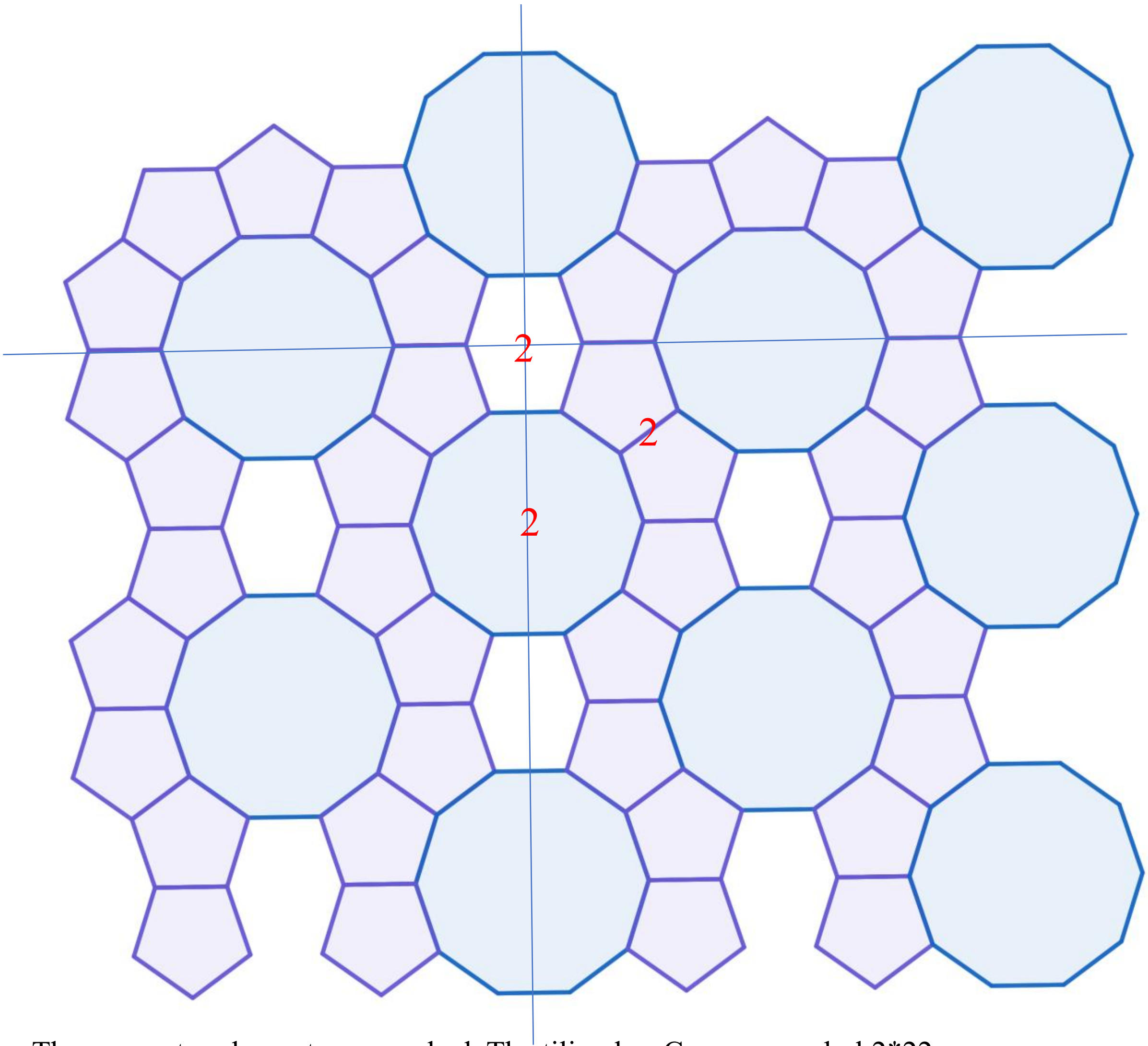
Here is the alternative construction diagram for what we notate 4.(5).(20). If we only count square vertices, we have a uniform tile pattern.



(6) Here is the expand-&-fill diagram for the tiling which includes the vertex (5,5,10):

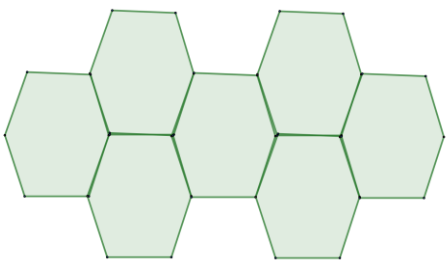


We shall call the green shape a *2*-form. Here is more of the tiling with the green shapes left uncoloured:

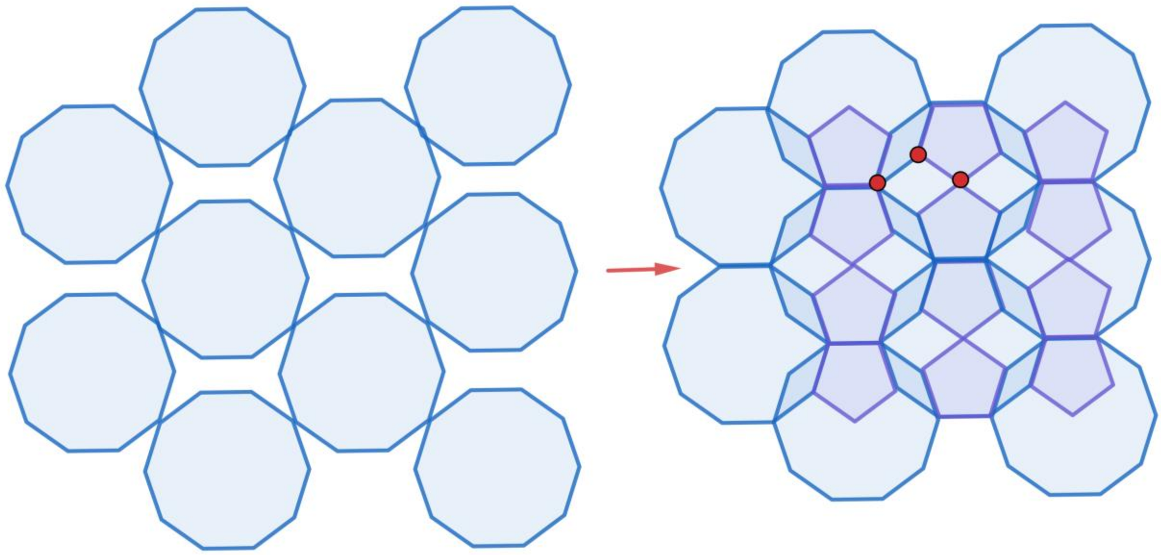


The symmetry elements are marked. The tiling has Conway symbol 2*22.

The *2*-form tiles the plane on its own:



An alternative construction is not as satisfactory in this case. All the polygons are covered. The symbol is (5).(5).(10). The vertices are of the three kinds marked. Considered as a tiling, we have three shapes: the regular pentagon and a fat and a thin rhombus, the two 'Penrose' rhombuses in fact. The Conway symbol is again, as it must be, 2*22. The tile pattern is 3-uniform.



General comment

By virtue of the means of construction, infill tilings have the symmetry of the seed.

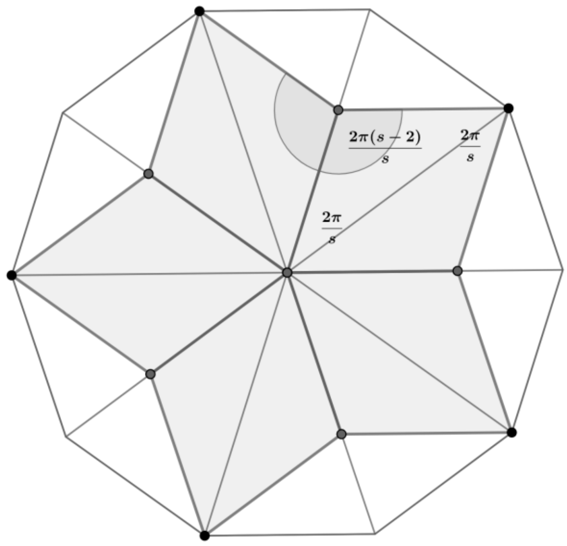
The irregular infill tilings: definitions and notations

Reviewing all our tilings of types D and E, it will be useful to classify and list the irregular polygons we have used. There are just three types. (We give definitions beneath the table.)

Rhombus	$\theta, \theta' \ \ \theta' = \pi - \theta$	$\frac{8\pi}{21}, \frac{13\pi}{21}$
Rhombic k-star	$k = 1$	$3_{\pi/9}$
	$k = 2$	$4_{\pi/10, 2\pi/5}$
k-form	$k = 2$	$2_{5,10}$
	$k = 3$	$3_{7,42}$
		$3_{8,24}$
		$3_{9,18}$
	$k = 6$	$6_{10,15}$

Kepler stars are defined as follows. The star m_α is an equilateral $2m$-gon with just two angles, that of the 'point', α, and that of the 'dent', the reflex angle $\frac{2(m-1)\pi}{m} - \alpha$. (The dent is not counted as a vertex unless shared by two other polygons.)

We take s rhombuses whose smaller angle is a unit fraction of a whole turn, $\frac{2\pi}{s}$, and arrange them round a point so that their smaller angles coincide:



The result is a Kepler star with $\alpha = \frac{2\pi}{s}$, where s is an integer > 2.

In other words, a rhombic star is a Kepler star whose point angle is a unit fraction of a whole angle.

A ***rhombic k-star*** is the union of the perimeters of k rhombic stars of common m, $k \geq 1$, dissected from the originals. It is notated $m_{\alpha_1,\alpha_2,\ldots}$

A ***k-form*** is a polygon whose (non-reflex) interior angles form a subset of those of the regular polygons sharing edges with it. It has k symmetry axes. It is notated k_{n_1,n_2} ... where the surrounding regular polygons are $n_1$gons, $n_2$gons, ... On this definition, considering the tilings $3^6, 6^3$ respectively, {3} is a *3*-form, {6} a *6*-form. Where 3 polygons meet at each vertex, the case with the 6 irregular infill tilings, that belonging to the *k*-form is the reflex of the sum of the other two, which is simply the third. In each case, regular or irregular, the subset consists of the interior angles of the two largest polygons of the three. In our case this is because the smallest polygon occurs in every other position with the result that the reflex angle always excludes it.

When we notate a vertex which includes an irregular polygon, we specify which angle it contributes. For example:

$\frac{8\pi}{21}$ signifies the smaller angle of the rhombus with angles $\frac{8\pi}{21}, \frac{13\pi}{21}$.

$4_{\pi/10}$ signifies the point angle of the contributing star with that symbol.

3_{42} signifies the interior angle in the *k*-form $3_{7,42}$ which would be the interior angle of the regular 42-gon.

(3,8,24)/(24,8,3) signifies a vertex occurring in both the clockwise and anticlockwise senses.

We can now fully specify our tilings by listing the vertex types:

(3,7,42/42,7,3; $3,3_7 42$; $3, 3_{42}, 7$; $3,42, 3_7$; $3, \frac{8\pi}{21}, 3,42$; $3, \frac{13\pi}{21}, 3,7$).
and the derived tiling $(3,3_7, 3_{42})/(3_{42}, 3_7, 3)$.

$(3,8,24/24,8,3;\ \ 3,3_{24},8;\ \ 3,24,3_8)$.
and the derived tilting $(3,3_8,3_{24})/(3_{24},3_8,3)$.

$(3,9,18/18,9,3;\ \ 3,9,3_{18};\ \ 3,3_9,18;\ \ 3,9,3_{\pi/9},9)$.
and the derived tilings $(3,3_9,3_{18})/(3_{18},3_9,3)$; $(3,9,3_{\pi/9},9)$.

$(3,10,15/15,10,3;\ \ 3,15,6_{10};\ \ 3,10,6_{15})$.
By convention we only consider angles where 3 edges meet, so the reflex angles 6_{r10}, 6_{r15} are not included, as already noted for Kepler stars..

$(5,5,10;\ \ 5,5,2_{10};\ \ 5,2_{10},10)$.

$(4,5,20/20,5,4;\ \ 4,4_{\pi/10},4,20;\ \ 4,4_{2\pi/5},4,5)$.

References

Stephens M., Stephens J., Parramore K., 2022, 'Fitting regular polygons round a point", *Mathematics in School*, **51**, 4, pp. 30 – 31.

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