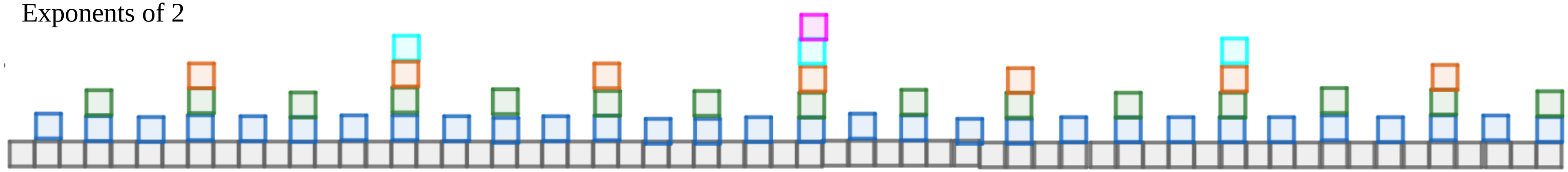
Number City: An advanced topic

How big is 60! ?

The factorial of a number n - written $n!$ - is the product of all the natural numbers up to and including n. We can imagine stacking all the blocks in each row of our model down at one end. Legendre's formula tells us how many we shall have for each prime: The power e_p of prime p in $n!$ = $\sum_{i=1}^{i=\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$. (The brackets mean 'the biggest integer smaller than the quotient', the *floor* function of the number.)

$$e_2 = \left\lfloor \frac{60}{2} \right\rfloor + \left\lfloor \frac{60}{4} \right\rfloor + \left\lfloor \frac{60}{8} \right\rfloor + \left\lfloor \frac{60}{16} \right\rfloor + \left\lfloor \frac{60}{32} \right\rfloor = 56.$$

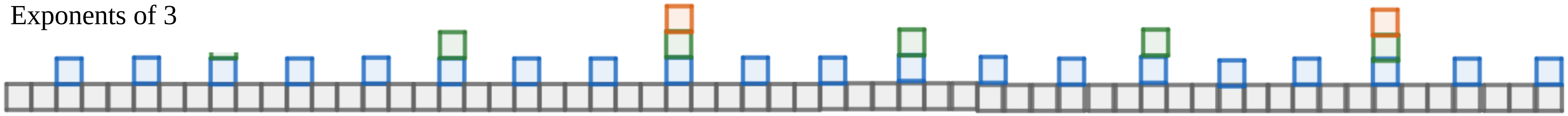
Exponents of 2



1 2 3 ...

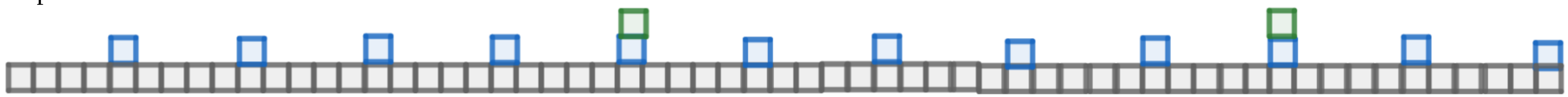
$$e_3 = \left\lfloor \frac{60}{3} \right\rfloor + \left\lfloor \frac{60}{9} \right\rfloor + \left\lfloor \frac{60}{27} \right\rfloor = 28.$$

Exponents of 3



$$e_5 = \left\lfloor \frac{60}{5} \right\rfloor + \left\lfloor \frac{60}{25} \right\rfloor = 14.$$

Exponents of 5



We find the powers of the other primes in the same way, giving the final total of

$2^{56} \times 3^{28} \times 5^{14} \times 7^9 \times 11^5 \times 13^4 \times 17^3 \times 19^3 \times 23^2 \times 29^2 \times 31 \times 37 \times 41 \times 43 \times 47 \times 53 \times 59.$

My calculator can't cope with that. But we'll aim for an order of magnitude estimate. We'll split the product into blocks and write each block in scientific notation. Because of the rounding error involved, we'll work out the figure in two ways: first from our Legendre calculation, second by simply multiplying out the consecutive numbers in the factorial.

$(2^{56}) \times (3^{28}) \times (5^{14}) \times (7^9) \times (11^5) \times (13^4) \times (17^3 \times 19^3) \times (23^2 \times 29^2) \times (31 \times 37 \times 41 \times 43 \times 47 \times 53 \times 59)$
$\approx (7.3 \times 10^{16}) \times (2.3 \times 10^{13}) \times (6.1 \times 10^9) \times (4.0 \times 10^7) \times (1.6 \times 10^5) \times (2.9 \times 10^4) \times (3.4 \times 10^7) \times (4.4 \times 10^5) \times (3.1 \times 10^{11})$
$\approx (7.3 \times 2.3 \times 6.1 \times 4.0 \times 1.6 \times 2.9 \times 3.4 \times 4.4 \times 3.1) \times 10^{(16+13+9+7+5+4+7+5+11)} \approx (8.8 \times 10^4) \times 10^{77} \approx 10^{82}.$

$1 \times 2 \times 3 \times \ldots \times 60 = (1 \times \ldots \times 15) \times (16 \times \ldots \times 25) \times (26 \times \ldots \times 34) \times (35 \times \ldots \times 42) \times (43 \ldots \times 50) \times (51 \times \ldots \times 56) \times$
$(57 \times \ldots \times 60) \approx (1.3 \times 10^{12}) \times (1.2 \times 10^{13}) \times (1.9 \times 10^{13}) \times (4.8 \times 10^{12}) \times (2.2 \times 10^{13}) \times (2.3 \times 10^{10}) \times (1.2 \times 10^7)$
$\approx (1.3 \times 1.2 \times 1.9 \times 4.8 \times 2.2 \times 2.3 \times 1.2) \times 10^{(12+13+13+12+13+10+7)} \approx (8.6 \times 10^1) \times 10^{80} \approx 10^{82}.$

So there we are, 60! is rather under 10^{82}.

Returning to Legendre's formula, if *n* is a power *m* of the prime *p*, the formula reduces to the sum of a geometric series with 1 for the first term, *p* for the common ratio, and *m* terms. So $e_p = \frac{p^m - 1}{p-1}$. If our model had continued to 64 = 2^6, e_2 would therefore have been $\frac{2^6-1}{2-1} = 63$.
(If we extend the model in our minds, we have to add to our 56, 1 for 62 and 6 for 64, total $56 + 1 + 6 = 63$, as predicted.)

If $n < p^m$, as 60 < 64, e_p will be $< \frac{n-1}{p-1}$. But consider two primes *p* and *q* and the ratio $\frac{e_p}{e_q}$ between them.
As $n \to \infty$, $\frac{e_p}{e_q} \to \frac{n/_{p-1}}{n/_{q-1}} = \frac{q-1}{p-1}$. If $p = 2, q = 3$, $\frac{q-1}{p-1} = \frac{3-1}{2-1} = 2$. Even for the low number 60, you see that we got $\frac{56}{28}$, exactly 2.
If $p = 2, q = 5$, the predicted ratio is $\frac{5-1}{2-1} = 4$, and we got $\frac{56}{14}$, spot on again! ***Try*** $\boldsymbol{p = 7, q = 19; p = 5; q = 29.}$

But you may notice I've been very selective in those examples. Do 'Legendre' calculations of the kind we did at the start for numbers a lot smaller and a lot bigger than 60 and find how the ratios work out.