4

The Rolling Parabola

www.magicmathworks.org/geomlab4

A) Roll the parabolic cut-out along the track, marking positions of the focus. Hang the chain in front of the trace to confirm that the curve is a catenary.

The geomlab file contains a proof.

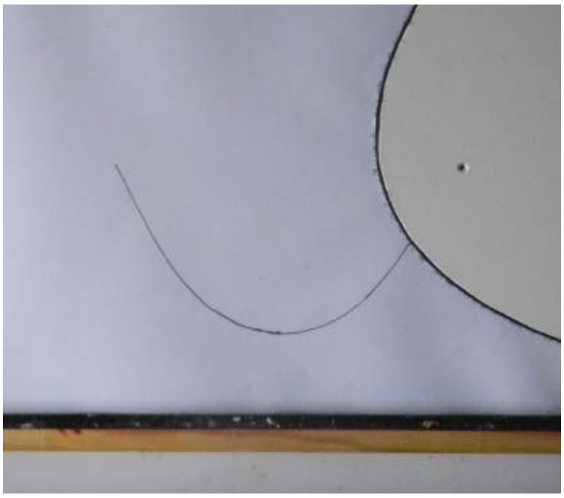
Galileo observed that the chain curve resembled a parabola near the vertex. It was only identified 27 years after his death.

B) Compare the 3 suspension bridges.

Where the decking is light compared with the cable, the curve approximates a catenary**;** where heavy, a parabola; between those extremes, some weighted hybrid. But, to the eye, there's little difference between any of them.

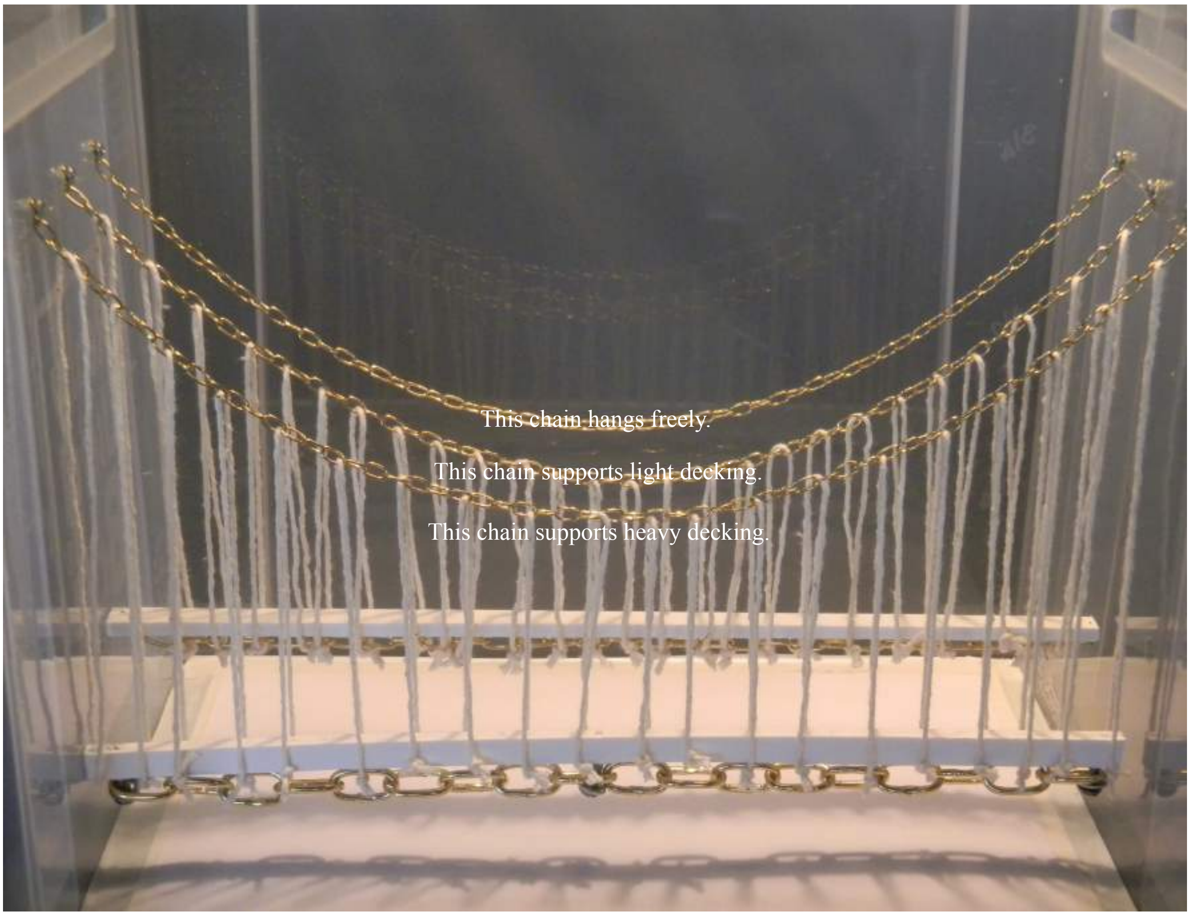
Paraboloid
Hyperbola
Hyperboloid
Hyperbolic paraboloid
Line pair
Sine curve
Tractrix
Exponential curve
Catenary
Catenoid
Helix
Helicoid
Plane
Polygon
Polyhedron
Tiling
Archimedean spiral
Equiangular spiral
Loxodrome
Line family
Circle
Cylinder
Cone
Sphere
Ellipse
Line
Parabola

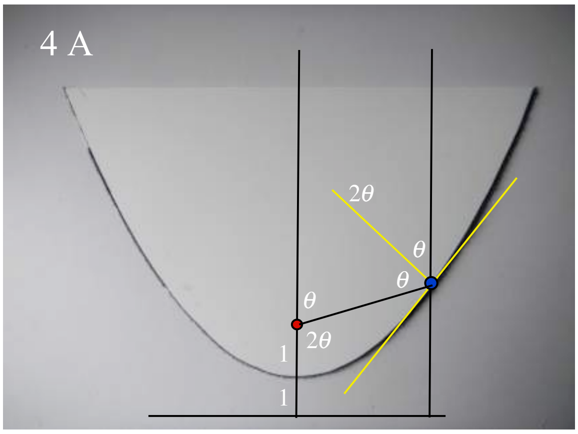
4 A

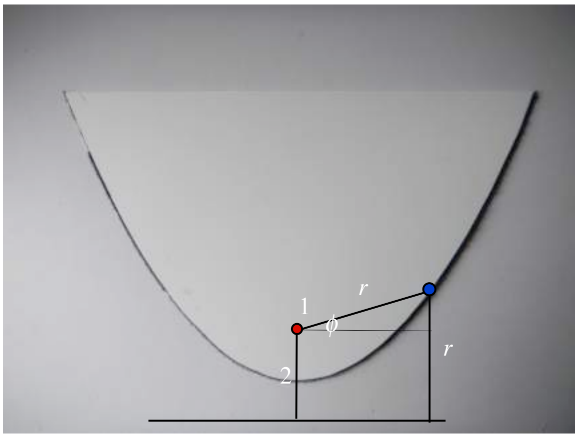


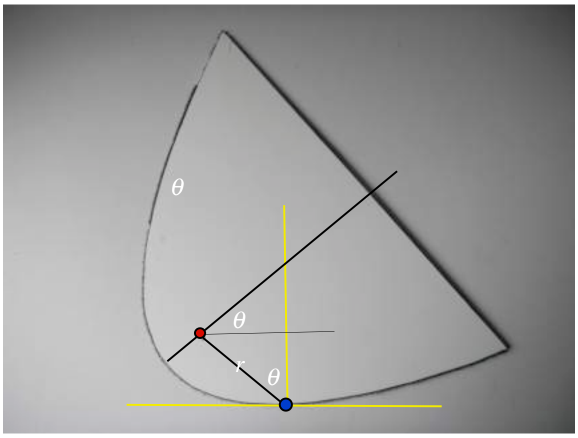

The hanging chain
matches the trace
exactly.

4 B


This chain hangs freely.
This chain supports light decking.
This chain supports heavy decking.







The top two pictures display facts about the parabola.

From the middle picture:

$\phi = 2\theta - \pi/2$, whence $r = 2 + r\sin(2\theta - \pi/2)$,
leading to $r = \sec^2\theta$. **(1)**

In the bottom picture the yellow lines have become our coordinate axes with the blue point as origin. This is our instantaneous centre of rotation. The red point is moving clockwise along the parabola's symmetry axis.

$$\frac{dy}{dx} = \tan\theta . \qquad \textbf{(2)}$$

$$y = r\cos\theta . \qquad \textbf{(3)}$$

Substituting **(1)**, **(2)** in **(3)**, $y = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$. **(4)**

$$\cosh u = \sqrt{1 + \sinh^2 u} = \sqrt{1 + \left(\frac{d(\cosh u)}{du}\right)^2} . \qquad \textbf{(5)}$$

Comparing **(4)** & **(5)**, $y = \cosh x$.

B

Here are the Taylor series expansions for the catenary and corresponding parabola:

Parabola: $f(x) = \frac{x^2}{2} + 1$ Taylor series: $1 + \frac{x^2}{2!}$

Catenary: $g(x) = \frac{e^x + e^{-x}}{2}$ Taylor series: $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

As long as $x \ll 1$ the terms after the red line don't count for much.

x	*g(x) - f(x)*
0.01	0.000000002
0.02	0.000000007
0.03	0.000000032
0.04	0.000000105
0.05	0.000000263
0.06	0.000000542
0.07	0.000001000
0.08	0.000001708
0.09	0.000002733

Indeed, up to around $x = 0.3$, the two functions agree to the third decimal place:

x	*f(x)*	*g(x)*
0.1	1.005	1.005
0.2	1.020	1.020
0.3	1.045	1.045
0.4	1.080	1.081
0.5	1.125	1.128 ←
0.6	1.180	1.185
0.7	1.245	1.255
0.8	1.320	1.337
0.9	1.405	1.433