Paraboloid

Parabola

Hyperbola

Line

Hyperboloid

Ellipse

Hyperbolic paraboloid

Sphere

Line pair

Cone

Sine curve

Cylinder

Tractrix

Circle

Exponential curve

Line family

Catenary

Loxodrome

Catenoid

Helix

Equiangular spiral

Helicoid

Archimedean spiral

Plane

Tiling

Polygon

Polyhedron

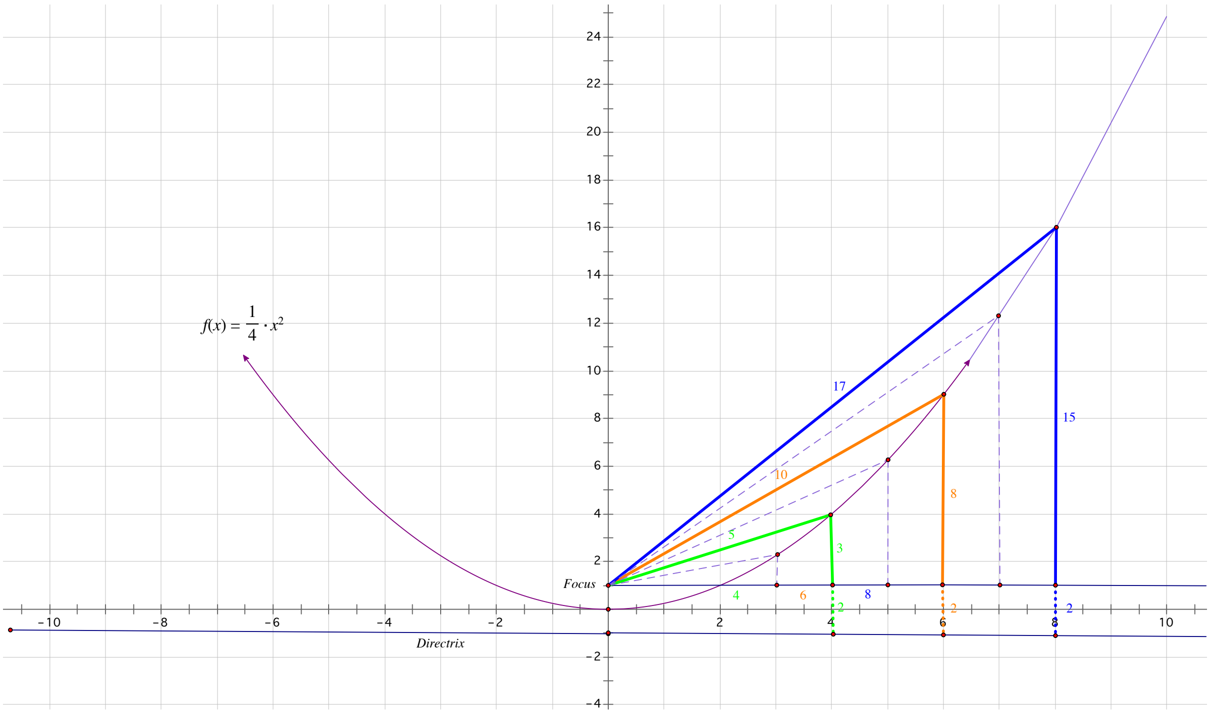
29

Parabola No. 10

www.magicmathworks.org/geomlab29

In **28** we produced the hyperbola with string. Here we do the same for the parabola. In **28** we maintained equal string lengths inside and outside the straw. Here we maintain equal string lengths from a point and a line, thus defining a parabola.

Keeping the string pressed against it, slide the set square left. Flip it over and slide it right.



The string construction produces every Pythagorean triangle in which the hypotenuse and another side differ by *d*, the distance between the focus and the directrix, therefore, by choosing every possible *d*, all the triples. Those generated may have to be scaled up or down to produce a primitive set, but all will be rational. What guarantees this is that, since integral *x* is chosen, the length of the vertical line is rational and so therefore is the length of the hypotenuse and the vertical side, (which differs from the hypotenuse by *d*). The parabola in question has the formula $f(x) = \frac{x^2}{2d}$. In the example shown $d = 2$. The solutions correspond to the Euclidean triple $(2mn, m^2 - n^2, m^2 + n^2)$ when $n = 1$. Proclus tells us that Pythagoras himself knew how to generate such cases.

On the graph the three dashed triangles need scaling to give a set of integers:

$\left(3, \frac{5}{4}, \frac{13}{4}\right) \rightarrow (12, 5, 13)$ $\quad$ $\left(5, \frac{21}{4}, \frac{29}{4}\right) \rightarrow (20, 21, 29)$ $\quad$ $\left(7, \frac{45}{4}, \frac{53}{4}\right) \rightarrow (28, 45, 53)$

The scale factor is 4 in each case, so the difference in length between the hypotenuse and the vertical side becomes 4 x 2 = 8.

The figure below shows that we can characterise the triple in terms of *d* like this:

$\left(x, \frac{x^2-d^2}{2d}, \frac{x^2+d^2}{2d}\right)$. Certain features emerge from the algebra:

If *d* is odd, and *x* is an odd multiple of it, we must scale the triple by $\frac{1}{d}$.
If *d* is even, and *x* is an odd multiple of it, the same applies.
If *d* is even, say *2a*, and *x* is an even multiple of it, we must scale the triple by $\frac{1}{a}$.

The triangles are duplicated for two reasons:

1) Since there are no isosceles Pythagorean triangles, every triangle appears twice: once for a *d* value representing the difference in length between the hypotenuse and the shortest side, and again for the difference between the hypotenuse and the middle side.

2) Due to scaling. For example, the brown triangle on the first graph records the triple (6, 8, 10). But this scales by ½ to the primitive triple (3, 4, 5), which is already present as the green triangle. And when we scaled up the dashed triangles we produced triangles which will be found again for the *d* values 1 & 8, 9 & 8, 15 & 8.