17
... and Back

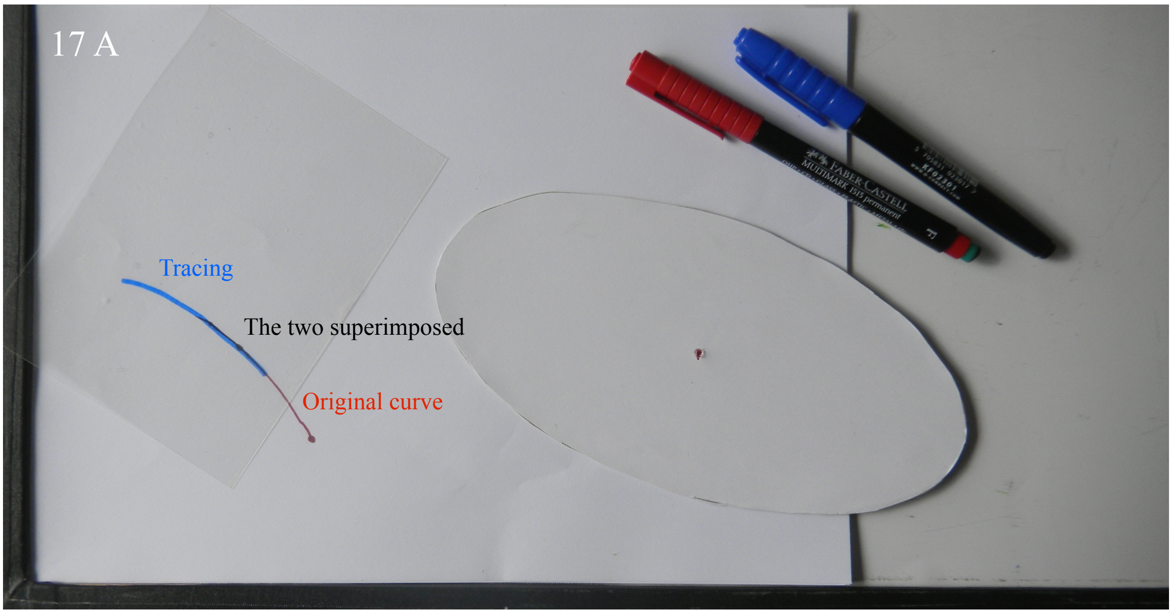
www.magicmathworks.org/geomlab17

A) Rest a pen in the centre of the elliptical 'ladder' and slide it down the 'wall'. Trace the curve on the clear plastic sheet and slide the trace along the curve, thus confirming that it is the arc of a circle. Repeat with the straight stick, a limiting case of the ellipse. Predict and confirm what happens with the outer holes. (Compare 'the trammel of Archimedes'.) Proof in the Geomlab file.

B) Find the ellipse pair *without* velcro. Set the loose one at right angles to the fixed one. Rest a pen in the hole and translate it round the fixed one.
Proof in the geomlab file.

C) Now take the pair *with* velcro. This time set them parallel with their minor axes aligned. Rest a pen at a focus of the loose one and roll it round the fixed one. The centre of the resulting circle is a focus of the fixed one. Proof in the Geomlab file.

Paraboloid
Hyperbola
Hyperboloid
Hyperbolic paraboloid
Line pair
Sine curve
Tractrix
Exponential curve
Catenary
Catenoid
Helix
Helicoid
Plane
Polygon
Polyhedron
Tiling
Archimedean spiral
Equiangular spiral
Loxodrome
Line family
Circle
Cylinder
Cone
Sphere
Ellipse
Line
Parabola

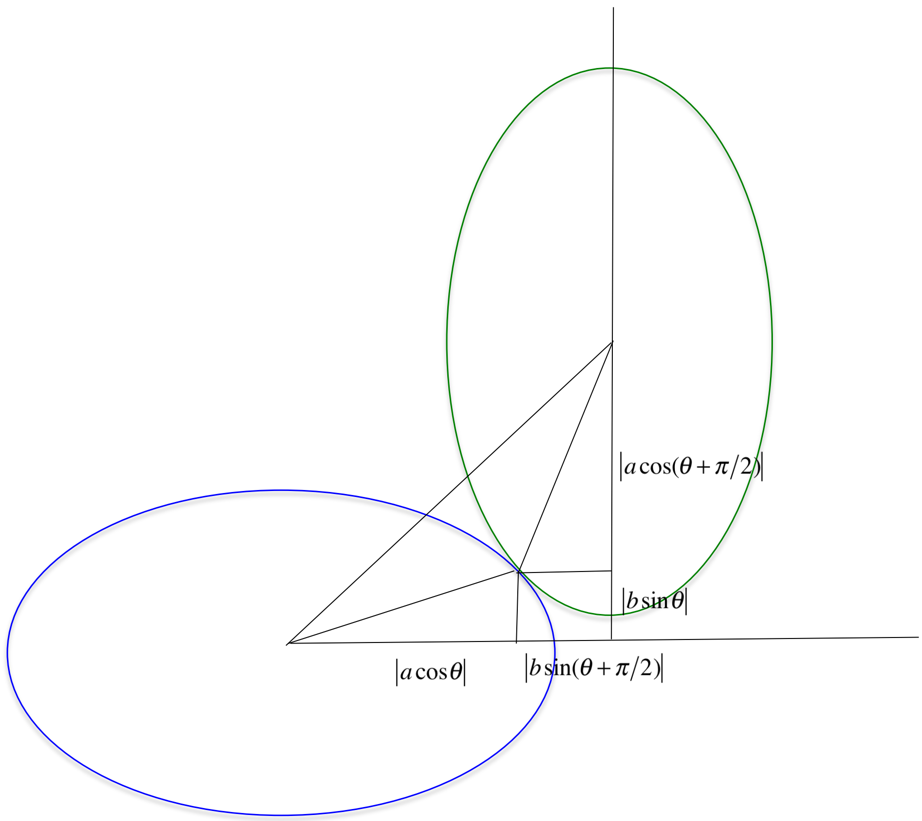

17 A
Tracing
The two superimposed
Original curve

17 A

We need the condition for a line to be tangent to an ellipse. We solve $y = mx + c$ with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to give a quadratic in x, then put in the condition for equal roots, obtaining $y = mx \pm \sqrt{b^2 + a^2 m^2}$. We square this to give a quadratic in *m*, representing a pair of tangents: $(x^2 - a^2)m^2 - 2xym + (y^2 - b^2) = 0$, then impose the condition for the two tangents to be perpendicular, i.e. for the product of the roots to be -1, and we have $x^2 + y^2 = a^2 + b^2$, the equation of a circle, centre the origin, radius $\sqrt{a^2 + b^2}$.

17 B

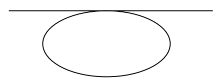
We apply Pythagoras' Theorem in this diagram:



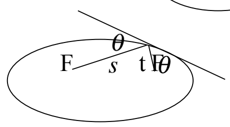
The locus is a circle of radius $a + b$ centred on the fixed ellipse. (N.B. θ is the angle measured at a point on the auxiliary circle of each ellipse. We are only using the fact that the angles for the two ellipses remain $\pi/2$ out of phase.) Note the algebraic symmetry. To read the vertical labels from the horizontal ones we just swap θ for $(\theta+\pi/2)$. There is an analogy in the plane with two congruent rectangles, sides s and t, set at right angles, where the locus is a square of side $(s+t)$.

17 C

At the start the figure is symmetrical about the common tangent:

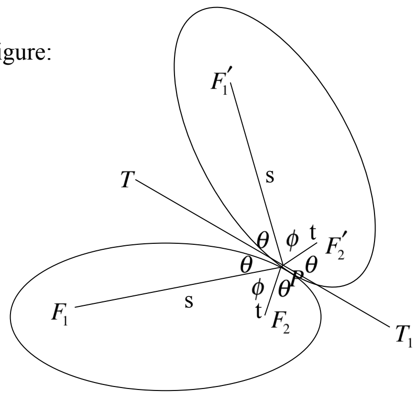


Because rolling is a symmetrical relation, it remains so.
To that fact we add two properties of the ellipse:



$s+t=2a$

to give this figure:



$4\theta+2\phi=2\pi$,
$2\theta+\phi=\pi$
$\Rightarrow F_1PF_2'$, F_2PF_1' straight lines,
with length $s+t=2a$
$\Rightarrow F_1'$ describes a circle centre F_2,
radius $2a$,
F_2' describes a circle centre F_1
likewise.