**15**

# Cylinder-Hyperboloid-Cone

**www.magicmathworks.org/geomlab15**

Find the model with strings joining two disks.

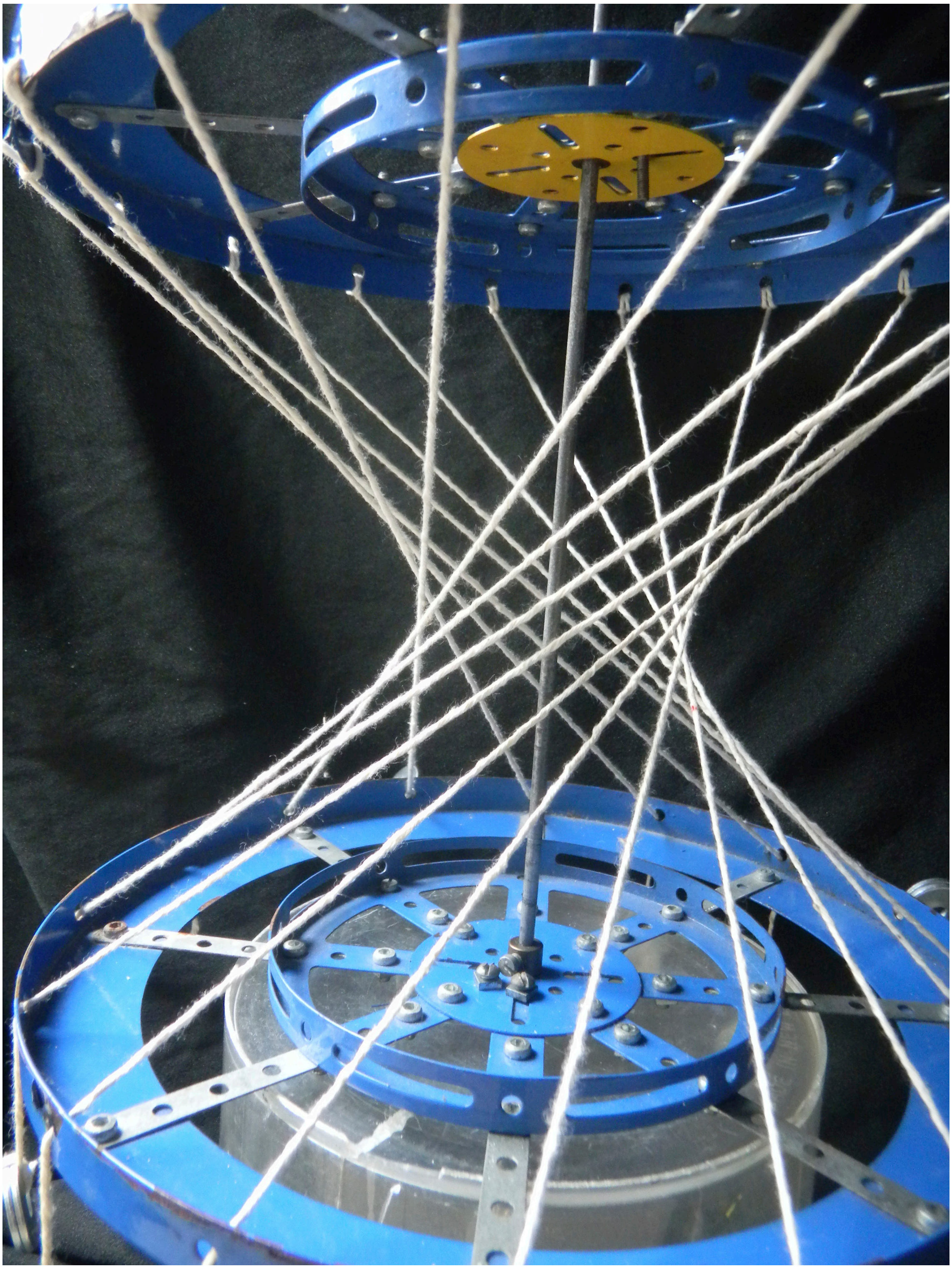
Begin with the strings vertical so that the resulting solid is a cylinder.

Rotate the top progressively to obtain in turn a hyperboloid ('of one sheet') and a double cone.

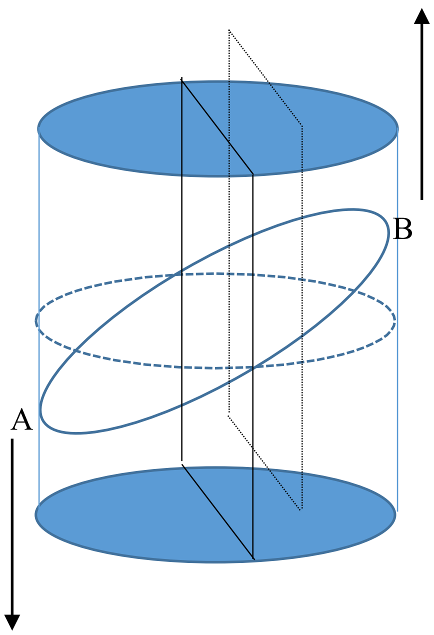
*The cylinder and cone are limiting cases of the hyperboloid, from which all the sections **14.1** to **14.4** can be obtained. Try experiment **7** again with the model where the hyperboloid is fixed in position and contains two spheres.*

Since both the cylinder and cone have elliptical sections, it must be possible to orientate the two so that their intersection has such a boundary. Study the models of success and failure. Proof in Geomlab file.

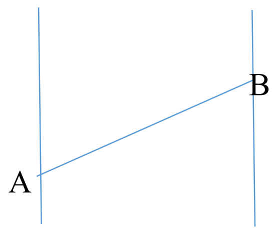
Paraboloid
Hyperbola
Hyperboloid
Hyperbolic paraboloid
Line pair
Sine curve
Tractrix
Exponential curve
Catenary
Catenoid
Helix
Helicoid
Plane
Polygon
Polyhedron
Tiling
Archimedean spiral
Equiangular spiral
Loxodrome
Line family
Circle
Cylinder
Cone
Sphere
Ellipse
Line
Parabola



To see why an oblique closed section of a hyperboloid of one sheet is an ellipse, we begin with the circular section of a cylinder and transform it in two stages. First we transform it into an oblique section of a cylinder. Then we take that and transform it into the section we want.

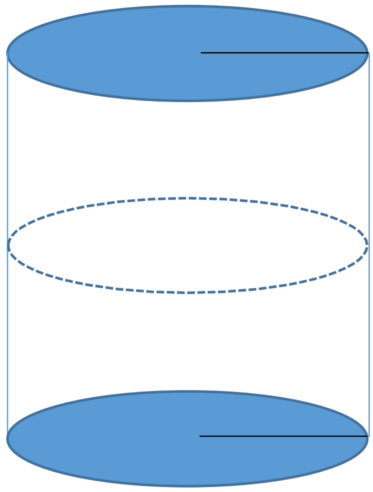


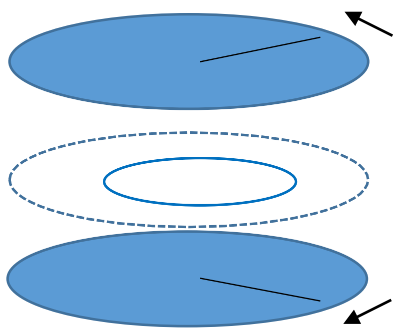

We shear the cylinder parallel to the black plane. The effect is to stretch the circle in the direction AB. The result is an ellipse. The projection perpendicular to the vertical plane through AB looks like this:



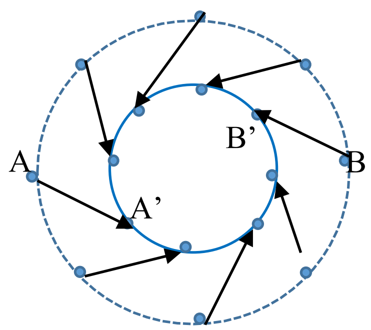

We therefore know that a curve which projects as a circle in plan and a straight line in elevation is an ellipse. We shall need this fact shortly.

We now locate points of the ellipse on the rulings which run parallel to the cylinder axis. We twist the top and bottom an equal amount in opposite senses relative to the central, horizontal plane, so that the rulings retain their original lengths:



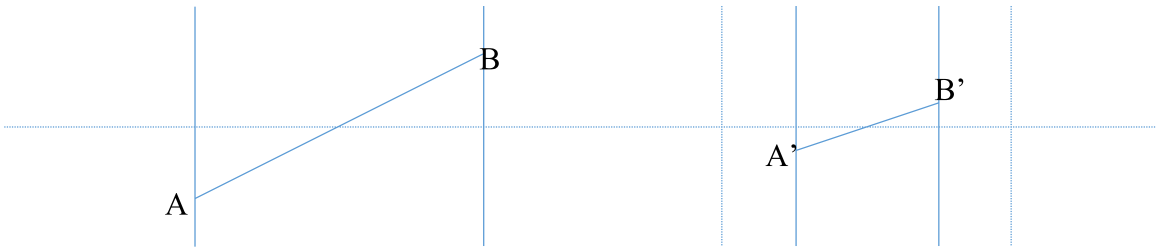


This is what happens in plan view:



The figure rotates and shrinks symmetrically. In plan, therefore, we still have a circle.

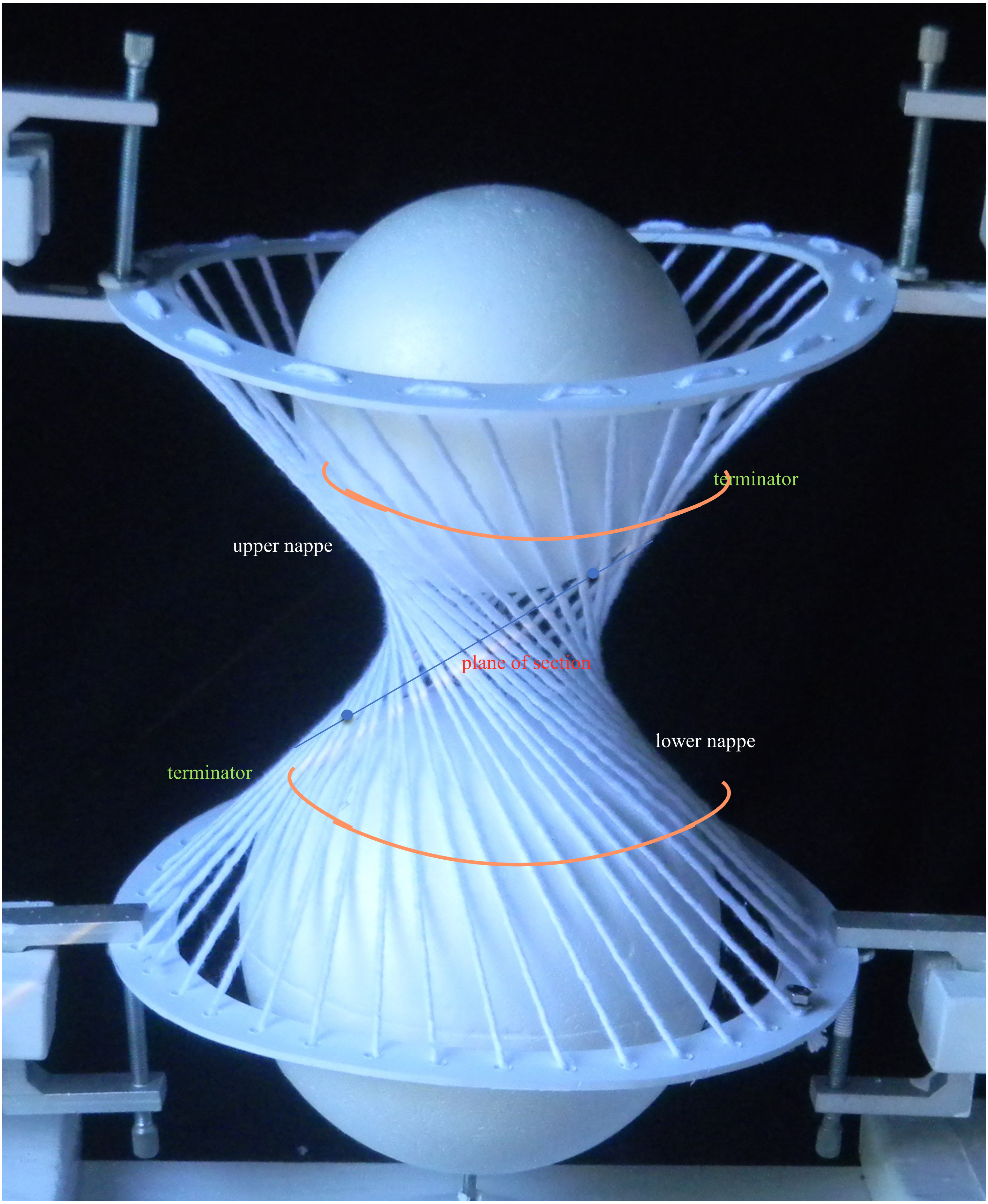
We compare projections onto the vertical planes through AB and A'B':




The figure has been uniformly compressed vertically so that the projection is still a straight line.

But we know that a figure which projects as a circle in plan and a straight line in elevation is an ellipse.

The 'Dandelin spheres' construction for the hyperboloid is the same as that for the cone and cylinder, which are just limiting cases of the hyperboloid: the plane of section touches the two spheres in the foci of an ellipse, but, as shown here, the ellipse can cut both *nappes*.



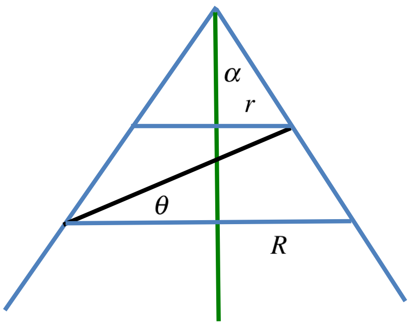
*Find the condition for a right circular cone of semiapical angle $\alpha$ and a right circular cylinder of radius $\rho$ to intersect in a plane section.*

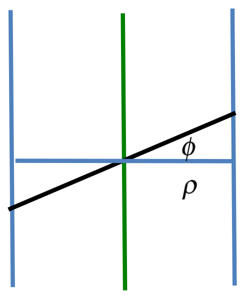
Any plane section of a cylinder which makes an angle > 0 with the axis is an ellipse. [**A**]
Any plane section of a cone which makes an angle > $\alpha$ with the axis is an ellipse. This condition implies that the plane cuts the axis. [**B**]

Thus the exercise reduces to achieving congruent ellipses, i.e. those with equal $a$ and $b$, (the lengths of the semimajor and semiminor axes respectively) in the two cases.

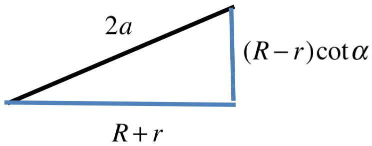
Choose any elliptical section of the cone, thus specifying $a$ and $b$, and also the radius $\rho$ of the cylinder, which must equal $b$. The ellipse and cone axes cannot be skew, for then a diameter of the cylinder would not correspond to the minor axis of the cone ellipse. Given that the cone and cylinder axes are coplanar, it remains only to determine the angle $\psi$ the cylinder axis makes with the cone axis.

Here are cross-sections through the axes of the respective figures. In the case of the cone the ellipse is tangent to two circular sections, one of radius $r$, the other of radius $R$.




$$b = \frac{r+R}{2} = \rho$$
$$\Rightarrow R = 2\rho - r \quad \textbf{(I)}$$




Applying Pythagoras' Theorem in the above triangle and substituting from (**I**) gives

$$(2a)^2 = (2\rho)^2 + (2\rho - 2r)^2 \cot^2\alpha \quad \textbf{(I1)}$$

$$2a = 2\rho\sec\phi \quad \textbf{(III)}$$

From (**II**) and (**III**) we have $\phi = \arctan\left[\left(\dfrac{\rho - r}{\rho}\right)\cot\alpha\right]$

But, using trigonometry in the above triangle and (**I**) again, we also have

$$\theta = \arctan\left[\left(\frac{\rho - r}{\rho}\right)\cot\alpha\right] = \phi$$

Since $\theta = \phi$, $\psi = 0$. Combining this condition with [**A**] and [**B**], we conclude that we require the axes of cone and cylinder to be parallel and the cone axis to fall within the cylinder.