FROM LATIN TO GRAECO-LATIN

Introduction

In a *Latin square of order k* each row and column contains all k elements. To put that in the negative, no element occurs twice in a row or column. The issue of *Symmetry* for summer 1993 discusses a famous example. And in the 20 years which have passed, a game based on Latin squares, Sudoku, has achieved unique popularity. In case you think there isn't much maths in Sudoku, go to http://nrich.maths.org, enter 'Sudoku' in the search box and you will find 43 different activities based on it. In case you're still not convinced, get a copy of this recently published book of 213 pages: 'Taking Sudoku Seriously: The math behind the world's most popular pencil puzzle' by Jason Rosenhouse and Laura Taalman.

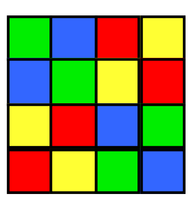
A *Graeco*-Latin square is a pair of Latin squares in which each pairing of corresponding elements is distinct. Thus in an order k Graeco-Latin square all k^2 possible pairings occur. Leonhard Euler was the first to study Graeco-Latin squares seriously. The name comes from his practice of using 'a, b, c, ...' for one set of elements and '$\alpha, \beta, \gamma, ...$' for the other.

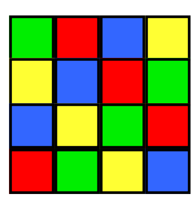
Finding these so-called *orthogonal* pairs makes a challenging puzzle when k = 4, (the first composite number). My starting-point for 'Graeco-Latin Squares' (*Symmetry+* issue 17, spring 2002) was the challenge 'Aunty's Teacups' in the NRICH hands-on maths roadshow, which I'd adopted in my own. I've therefore had the chance over several years to observe how students tackle the problem.

It has so many interesting aspects that I want to spend the next 7 sections 'unpacking' my original piece by going back to the Latin squares themselves (sections **1** to **4**) before assembling the orthogonal pairs (sections **5** to **7**).

From Latin to Graeco-Latin 1: Counting the different-*looking* Latin Squares of Order 4

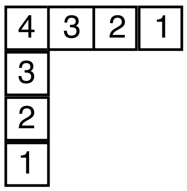
Here are 2 different-looking squares:





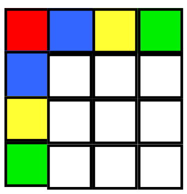
How many are there altogether?

Imagine we're building the square from the top left corner. We start with the top row.
We're free to choose the first cell in 4 ways, the second in 3, the third in 2, the fourth in 1. Then we work down the first column in the same way:



Here are the number of choices for each cell.
To find the number of possible arrays this shape, we just form the product (4 x 3 x 2 x 1) x (3 x 2 x 1) or 4! x 3! = 144.
(We read '4!' as '4 factorial' or, more dramatically, '4 shriek'.)

We fix one such array and find how many different ways we can arrange elements in the 3 x 3 square left:



Experiment with colours on squared paper. You should find there are just 4 ways to fill the rest of the grid.

If there are 144 different 'L'-frames
and for each there are 4 ways to complete the square,
there must be 4 x 144 = 576 different Latin squares of order 4.

Apply this method to Latin squares of order ***3***. How many are there of those?

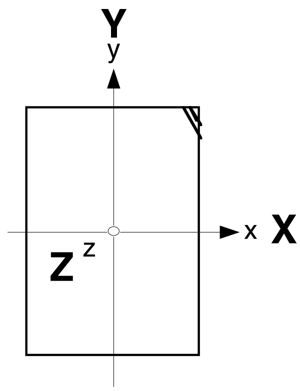
From Latin to Graeco-Latin 2: Defining Latin Squares

So, there are 576 different-*looking* Latin squares of order 4.
If I point to 2 of these and say, "Ah, but these are really the same", what do I mean?

I mean that, if they were tables displaying information, the information they displayed would be the same. But that too requires explanation:

Consider a particularly important Latin square of order 4. It records what happens when you take a rectangle of card and perform symmetry operations on it in sequence. (Those who read Martin Perkins' piece 'The Algebra of the Equilateral Triangle' in the issue of *Symmetry* for summer 1994 will be familiar with the ideas here.)

We take our card - slightly dog-eared so that we can keep track of what happens to it.
We align its symmetry elements with coordinate axes - 2 mirror lines (which we can think of as axes of half-turn symmetry lying in the plane of the card) with the x- and y-axes and an axis of half-turn symmetry with the z-axis:



Key to symmetry operations

I	do nothing
X	flip (rotate a half-turn) about the x-axis
Y	flip (rotate a half-turn) about the y-axis
Z	rotate a half-turn about the z-axis

We find that when we perform one such operation after the other, the result is simply one of the single operations in our set. We call such a special set a *group* . Our set is the *symmetry group of the rectangle* but the same pattern might relate many other sets of things: because we're *abstracting* the pattern from any particular context, we call it an *abstract group* . This one is called the *Klein four-group* after the mathematician Felix Klein.

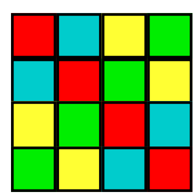
It is defined by the table which records the result of combining (or *composing*) operations. (By convention we perform the operation along the top followed by the one down the side). Check its correctness with a playing card:

	I	**X**	**Y**	**Z**
I	I	X	Y	Z
X	X	I	Z	Y
Y	Y	Z	I	X
Z	Z	Y	X	I

Notice that the table is a Latin square.

We can swap colours for the letter labels:

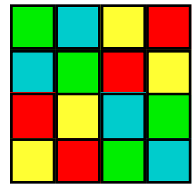
A



Now - and here is the point of this example - the way we arrange the row and column headings is purely a matter of convention. Let's permute them and observe the result:

	I	Y	X	Z
Z	Z	X	Y	I
X	X	Z	I	Y
I	I	Y	X	Z
Y	Y	I	Z	X

B



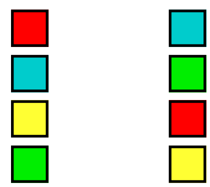
By our definition, A and B are the same.

Say we change our colour code:

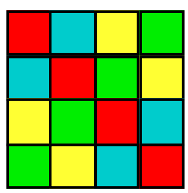
Key to symmetry operations

I do nothing
X flip (rotate a half-turn) about the x-axis
Y flip (rotate a half-turn) about the y-axis
Z rotate a half-turn about the z-axis

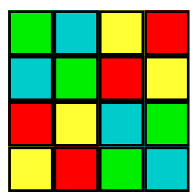
B coding B' coding



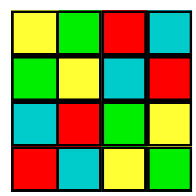
A



B



B'

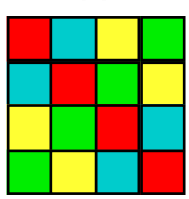


It makes no difference: A, B and B' are all the same square.

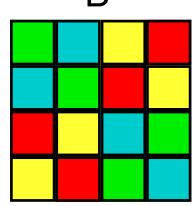
If we have two mathematical objects and the features of one exactly parallel those of the other, we call them *isomorphic* . A, B and B' are more closely related than that because one object is simply one of the others relabelled. We call them *automorphic* .

From Latin to Graeco-Latin 3: Distinguishing Latin Squares

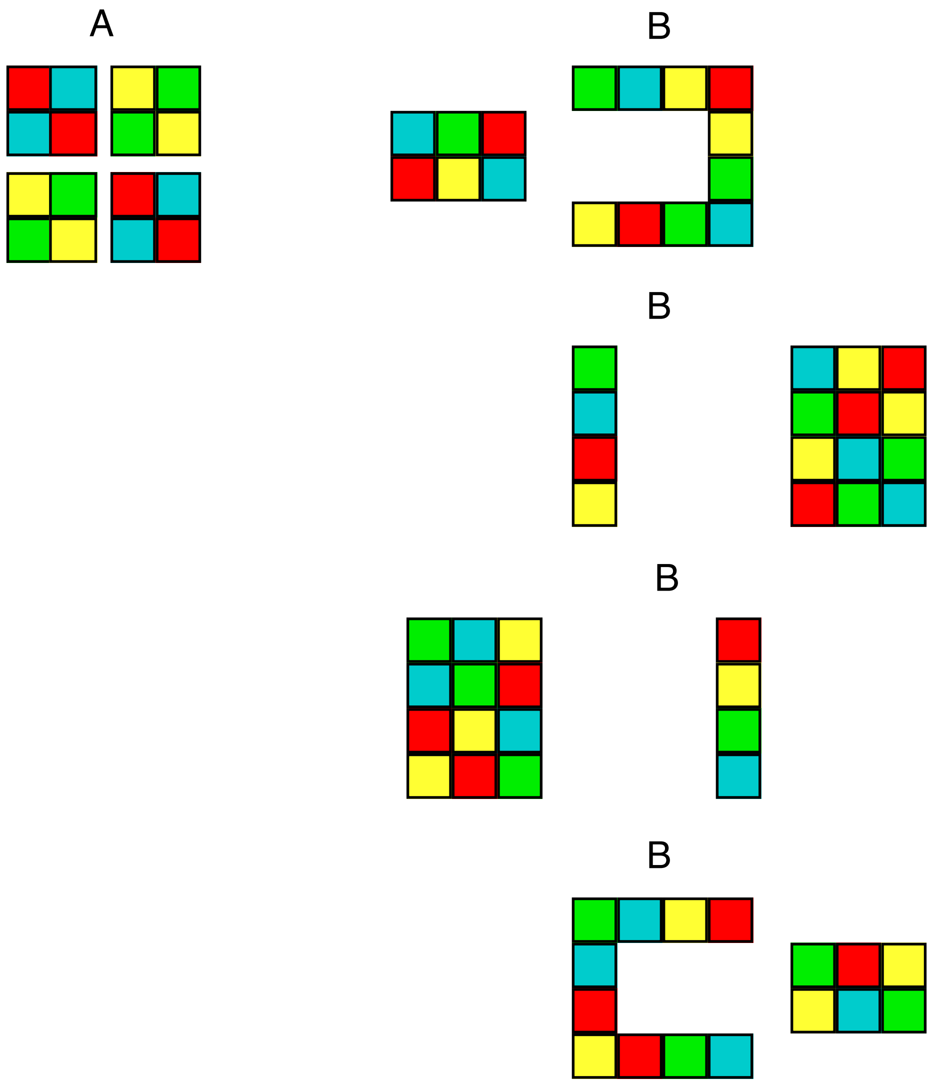
A



B



Notice that within A there are 4 order 2 Latin squares: 2 containing red and blue, 2 containing yellow and green. They don’t occur in B as Battenberg cakes but they do occur at the vertices of rectangles:



The effect of relabelling a square, as we did, is to swap rows and columns. When you swap rows and columns, you preserve order 2 Latin squares. The number of order 2 Latin squares is an *invariant*.

You’ll realise from our colour-coding change in section **2** that the squares would have been the same even if the colour-pairings on the right had differed.

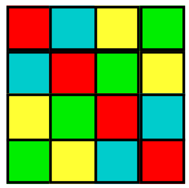
It’s very easy to disguise yourself if you’re a Latin square!

From Latin to Graeco-Latin 4: The Geometry of Latin Squares

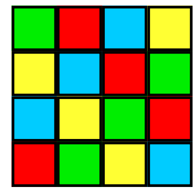
From an algebraic point of view, then, A and B are the same object.

Have a look at A and C. (C is the right-hand example shown at the start of section **1**.)

A



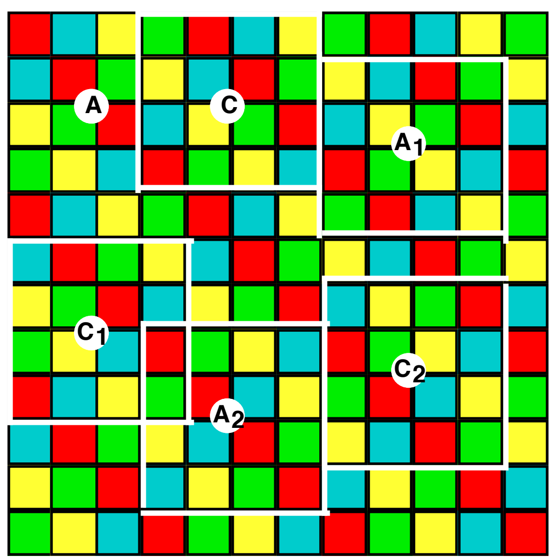
C



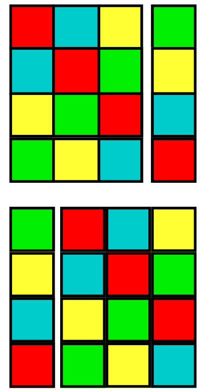
Glancing from one to the other, you're immediately aware of some connection - but what?

Imagine you 'block print' A to produce a tiling. You can find in it not only C but versions of A (A1, A2) and C (C1, C2) in different colour schemes:

An 'A' tiling



Concentrate on the original A and C. To get from A to C all we do is move all the cells one to the right, 'scrolling' the rightmost ones round to the left: we have performed a *cyclic* permutation:

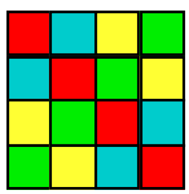


So, from an algebraic viewpoint, A, B and C are all the same.

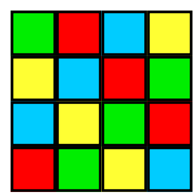
From a geometric viewpoint, so are A and C if we think of them as parts of a tiling.

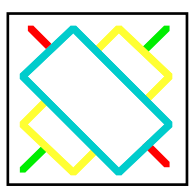
But, thought of as objects in their own right, they differ:

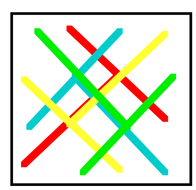
A



C







We consider like elements to form a motif. In A we have 2 lines and 2 oblongs. In C we have 4 (fat) Ts.

By my reckoning there are 8 motifs of this kind and they combine to give 13 different squares. For want of a better term, we'll call them the order 4 'Latin patterns'.

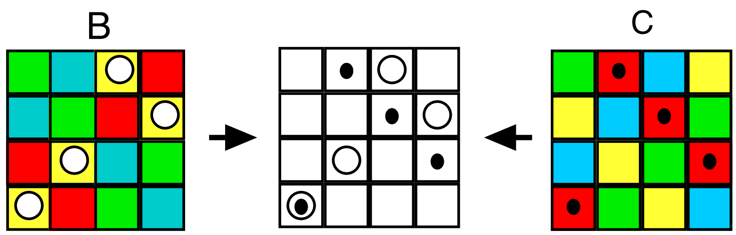
We can talk about the 'colour-blind' symmetries of the squares, i.e. their symmetries ignoring colour. A and C retain the full symmetries of a square:

- but others vary. Experiment with Multilink cubes on a pegboard or colours on squared paper. See if you can find my 13 and identify their symmetry elements.

From Latin to Graeco-Latin 5: Detecting Orthogonal Latin Squares from their 'Patterns'

By looking at the symmetries of 2 squares we can tell if they're compatible.

Each motif connects cells of the same colour. Since the cell pairs in the Graeco-Latin square are unique, motifs must overlap at exactly 1 point. This rules out A and C since a line, a rectangle and a (fat) T overlap either at 2 points or none. The same applies to A and B with its 4 (thin) Ts. But look at B and C. One consists entirely of fat Ts, the other of thin. A fat T and a thin T overlap in only 1 cell:



B and C are therefore orthogonal.

Try this approach on 2 of the other order 4 Latin patterns you have found.

From Latin to Graeco-Latin 6: Creating Orthogonal Latin Squares by arranging ready-made Cell Pairs

Because a Graeco-Latin square is a layer of 'cups' on a layer of 'saucers' there are two ways of building it:

a) Arrange all the saucers; arrange all the cups.
b) Make the cup-saucer pairs and shuffle them around.

In my original article I suggested building each cup-saucer pair as a tower of Multilink cubes on a Multilink pegboard. This allows either strategy. Some students choose one, some the other. Strategy (a) is generally the more successful. I suspect this is because you have a bird's-eye-view of the structure: with (b) you're trying to look in the x-, y- and z-directions all at the same time.

In the 'Multilink' embodiment the elements of the two Latin squares are of the same kind: coloured cubes. Students find the problem far easier if they're distinct: cups and saucers.
Toni Beardon (www.nrich.maths.uk/mathsf/journalf/sep02/art3) suggests using playing cards. Take the 16 court cards (jack, queen, king, ace). The 4 suits are your 'saucers' and the 4 values, your 'cups' (or vice versa!) Because the element-pairs are ready-made, you're forced to use strategy (b) but students find this embodiment far more engaging than my Multilink towers and generally achieve success more quickly.

What systems do they adopt?

Most students launch into this activity by adopting a scheme and attempting to follow it through. Duplication in rows and columns is out, so a common ploy is to make diagonal stripes. Carried through, this strategy results in failure so they back-track to a main diagonal of all the aces, say, and work from there.

In marked contrast, a year 7 Japanese student adopted the following strategy.

A Graeco-Latin square represents the most thorough mixing of the elements possible, their most disordered state - hence its importance in GM crop trials and the like (see part 2 of the original piece: 'Maths at Work'). This student decided to start at the opposite extreme, arranging the cards in their most highly ordered state - all the jacks down the first column, all the queens down the second, ... - and set out *systematically* to mix them up.

Lay out the 16 cards in this way and see how *you* would go about this.

The 4 x 4 square has one more challenge to offer.

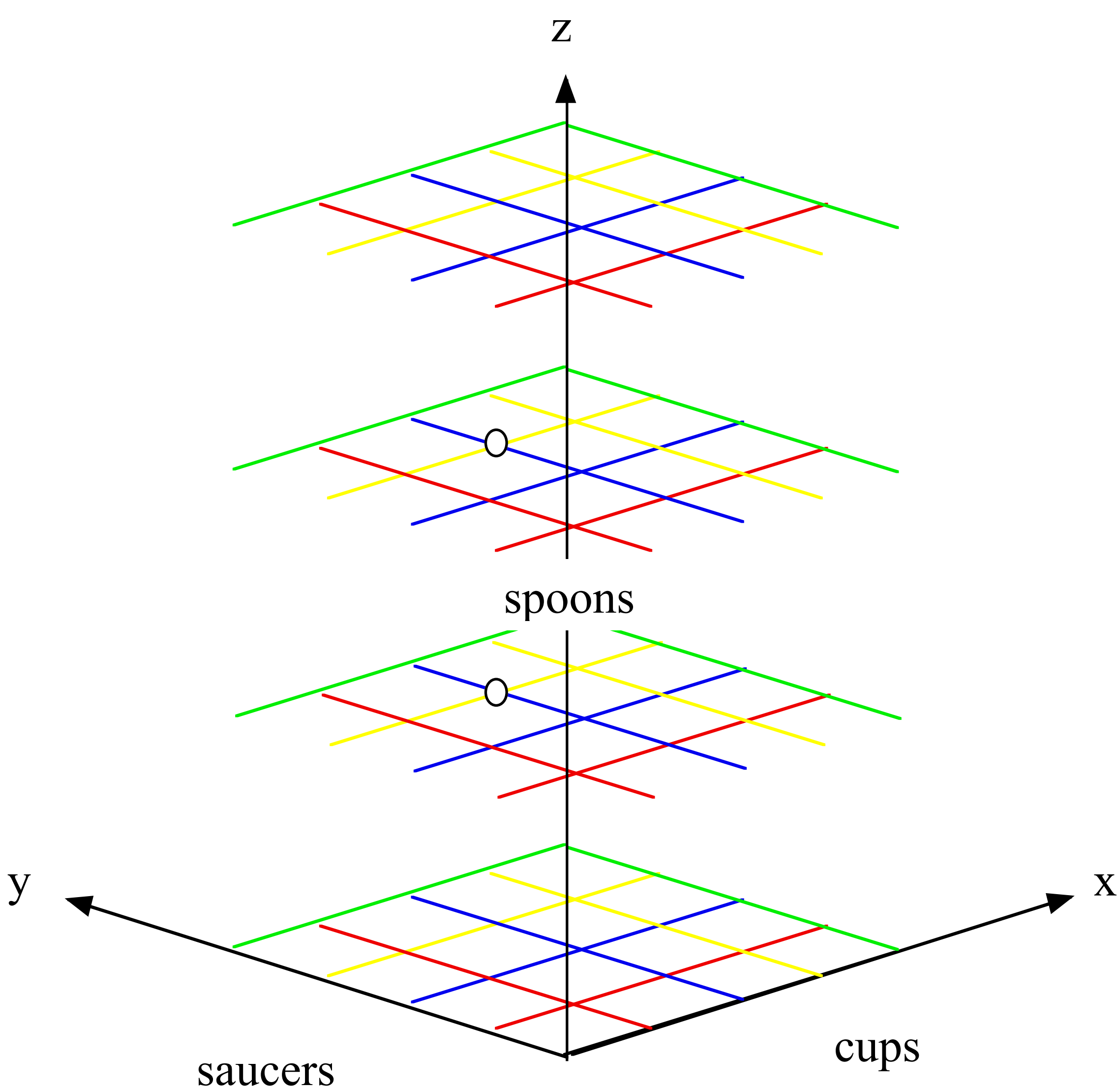
To your arrangement of cups and saucers you are now to add spoons. You must do so in such a way that, not only are the cup-&-saucer squares orthogonal, but so are the cup-&-spoon squares and the saucer-&-spoon squares.

Here is the cup-saucer-spoon *space*. You can imagine that the planes parallel to the x-y plane are modelled by perspex plates with holes drilled at grid points so that marbles can rest in them.
Each plane parallel to the x-z plane contains saucers of the same colour.
Each plane parallel to the y-z plane contains cups of the same colour.
Each plane parallel to the x-y plane contains spoons of the same colour.

Say you put marbles in the two holes shown. They lie in the line in which the planes ‘blue cups’, ‘yellow saucers’ cut, thus showing that you have duplicated a cup-saucer pair and failed to produce orthogonal squares. You have only achieved a solution when:

- No line parallel to an axis contains 2 marbles.
- Every plane contains 4 marbles.
- (Putting those two conditions together), every line contains 1 marble.

You should end up with 16 marbles altogether, one for each cup-saucer-spoon triplet.



From Latin to Graeco-Latin 7: Algorithms to generate Graeco-Latin Squares

Serious students, whether aged 10, 20 or 70, may ask themselves, “Well, that’s all very interesting: where does it lead?”

This last section is intended as a signpost.
We wish to construct an orthogonal pair of order 3.

In modulus (‘clock’) arithmetic the result of an operation is the remainder after division by the modulus. To achieve our aim we can use the integers modulo 3 as follows.

Here is the addition table for $(s + t)_3$:

$(s + t)_3$	t = 0	1	2
s = 0	0	1	2
1	1	2	0
2	2	0	1

Here is the addition table for $(2s + t)_3$:

$(2s + t)_3$	t = 0	1	2
s = 0	0	1	2
1	2	0	1
2	1	2	0

... and we have our Graeco-Latin square.

If we try to do the same thing for an orthogonal pair of order 4, however, we get into trouble:

$(s + t)_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$(2s + t)_4$	0	1	2	3
0	0	1	2	3
1	2	3	0	1
2	0	1	2	3
3	2	3	0	1

Our second square is not Latin.

When the modulus is prime, the integers behave like the rational numbers (integers and fractions) of ordinary arithmetic: you can add, subtract, multiply, divide and always get an answer - and one which is unique. Such a set under those operations is called a *field* . The field of rational numbers is infinite. Ours is finite. Finite fields are sometimes called *Galois* fields (GF) after their first investigator, Evariste Galois. (Use a search engine to learn the extraordinary end of this young man.) The integers modulo 3, then, form a field; those modulo 4 do not: we've just seen that, for example, 2(1) + 1 gives the same answer as 2(3) + 1. Galois, however, found a set of 4 elements which *do* constitute a field.

We take the integers modulo 2 and also the root of the equation $x^2 + x + 1 = 0$. Feed in 0 and 1 and you'll find there is none! But we imagine there is and give it a name, 'a' say.

This gives us some strange properties - recall that in binary arithmetic 1 + 1 = 0 and therefore that double anything is zero:

$a^2 + a + 1 = 0$. $a^2 + a + 1 = 0$.

Add 1 to both sides: $a^2 + a = a(a+1) = 1$. Add $(a+1)$ to both sides: $a^2 \qquad = a + 1$.

Check this addition table:

(s + t)2	t: o	1	a	a+1
s: o	o	1	a	a+1
1	1	o	a+1	a
a	a	a+1	o	1
a+1	a+1	a	1	o

Remember the Klein four-group? Here we have the same operation table - a nice example of an isomorphism.

To pull off our ‘Graeco-Latin’ trick using GF(4), instead of s + t and 2s + t, we must form
as + t and (a+1)s + t:

(as + t)2	t: o	1	a	a+1
s: o	o	1	a	a+1
1	a	a+1	o	1
a	a+1	a	1	o
a+1	1	o	a+1	a

((a+1)s + t)2	t: o	1	a	a+1
s: o	o	1	a	a+1
1	a+1	a	1	o
a	1	o	a+1	a
a+1	a	a+1	o	1

... and there’s our orthogonal pair.

In this section we’ve used a sledge hammer to crack the proverbial nut. However, we’ve also shown how you can move between two areas of mathematics - here, geometry and algebra. We’ve also introduced more ‘powerful’ mathematics, that is to say, mathematics we can apply to a wider range of problems.

Paul Stephenson
The Magic Mathworks Travelling Circus