Finding the one stone

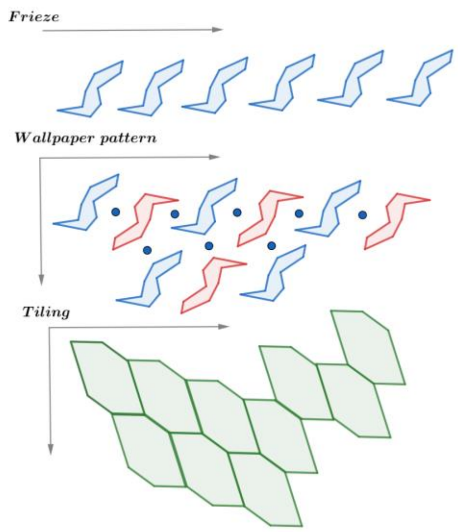
In the following notes the students are addressed directly, the teacher in brackets [].

[The students follow the story of the discovery of the first aperiodic monotile.
The mathematics is centred on the inflation of a tiling by repeated application of **substitution rules**, for that provides one method by which aperiodicity can be established. New terms are listed on a flipchart as they are introduced. A laser pointer is needed to point out features on the slides.]

This workshop is about a major discovery made this spring by an amateur mathematician but it's going to take us a while to get to it.

Part 1: Defining terms

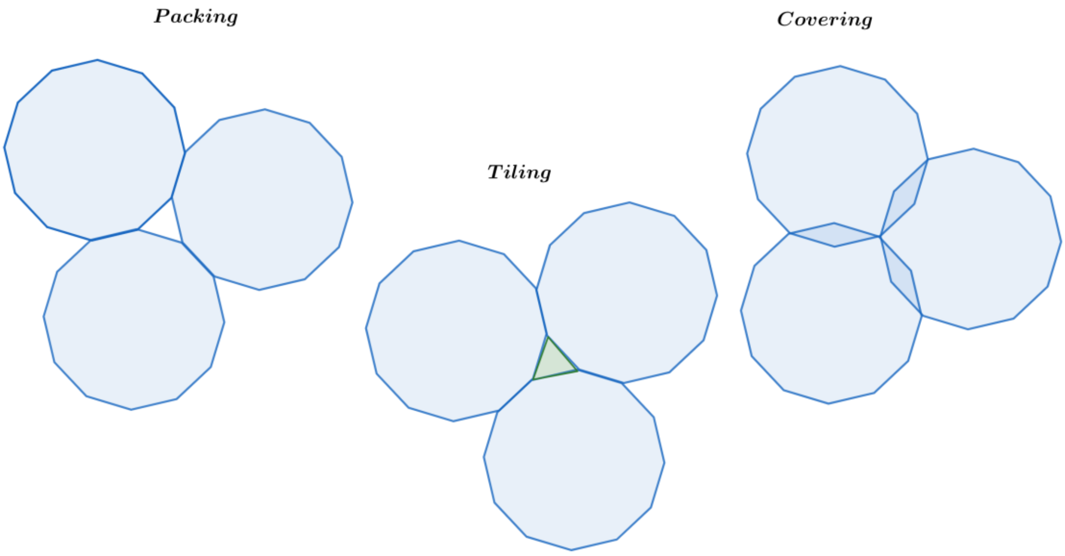
Slide 1



What do the three designs have in common? [They all have **translational symmetry**.]
Another way to put that is to say the designs are **periodic**.

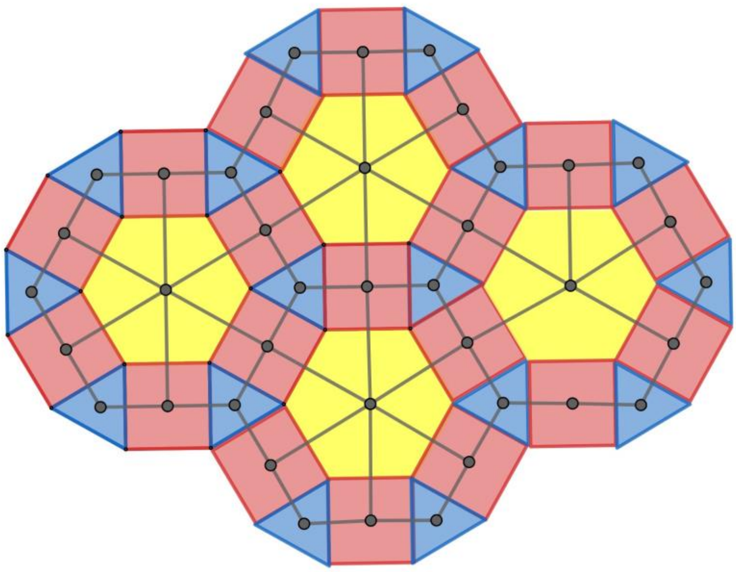
Slide 2

[Definitions of **tilings** v. packings, coverings, tilings as the 'Goldilocks' case]



E1 Arrange the equilateral triangles, squares and regular hexagons so that every **vertex** is identical, that is to say, round each vertex you meet the same shapes in the same order. We say that such a tiling made of regular polygons is **Archimedean.**

Slide 3



3.4.6.4 The **incentres** of the tiles you've just arranged are joined to make the sides of a new tiling. The original tiling and the new one are **duals**. A dual relation is a reciprocal one. If you join the incentres of the coloured tiles, you get the black-edged ones; if you join the incentres of the black-edged tiles, you get the coloured ones. This particular dual tiling is going to be important later.

Part 2: Substitution rules

What we are after in this workshop are tilings which are not only non-Archimedean but not periodic at all: **aperiodic** tilings – the prefix **a** means 'not. But there's a snag there. If I show you some tiles and claim that they only make tilings which are nperiodic, you are going to ask me how I can know. Can I **prove** that they make an aperiodic tiling? This workshop is about one method for doing so. It involves **substitution rules.** Before I say why these are involved, we'll look at a number of examples not necessarily having anything to do with tilings.

Slide 4

Inflation of a binary symbol

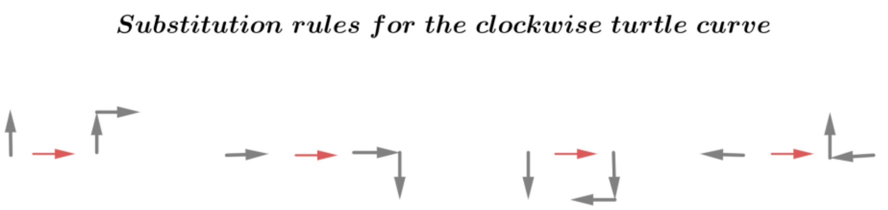
Substitution rules

$1 \rightarrow 10$
$0 \rightarrow 01$

		1
Iteration	1	1 0
	2	1 0 0 1
	3	1 0 0 1 0 1 1 0
	4	1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1
	5	1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0

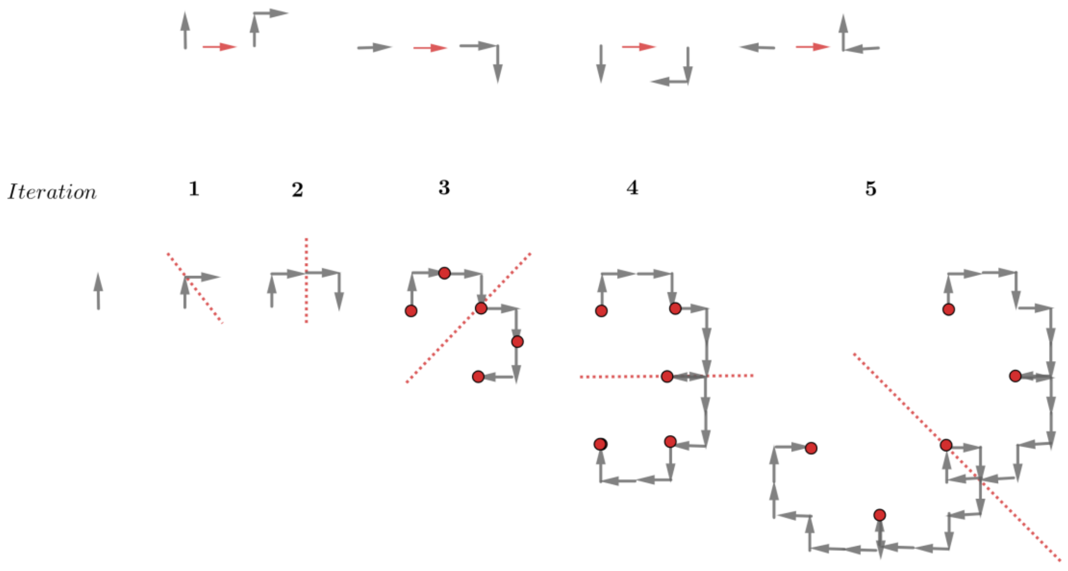
There are several new terms on this slide. But perhaps you can work out what's happening? We have a pattern: **inflation** suggests that the symbol string is growing in length; **iteration** suggests we keep repeating the same operation. The operation is to **substitute** '10' every time we see '1' and '01' every time we see '0'. Note that the total number of 0s and 1s doubles with each iteration. Can anyone spot the difference between lines 1, 3 & 5 and lines 2 & 4? [Lines 1, 3, 5 are palindromes; in lines 4 & 6, a '1' reflects as a '0' and vice versa.]

E2 Slide 5



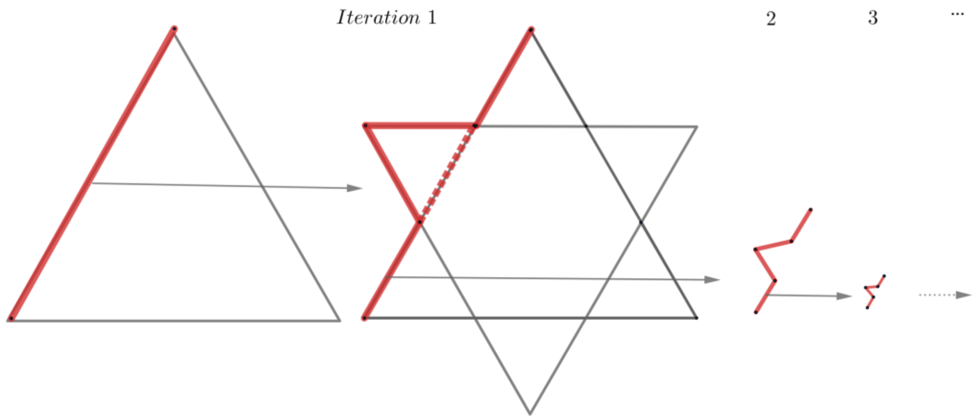
Using the squared paper, follow the substitution rules through the first few iterations. As you see, what they do in effect is to advance the turtle one step, then turn it clockwise and advance it another step.

Slide 6



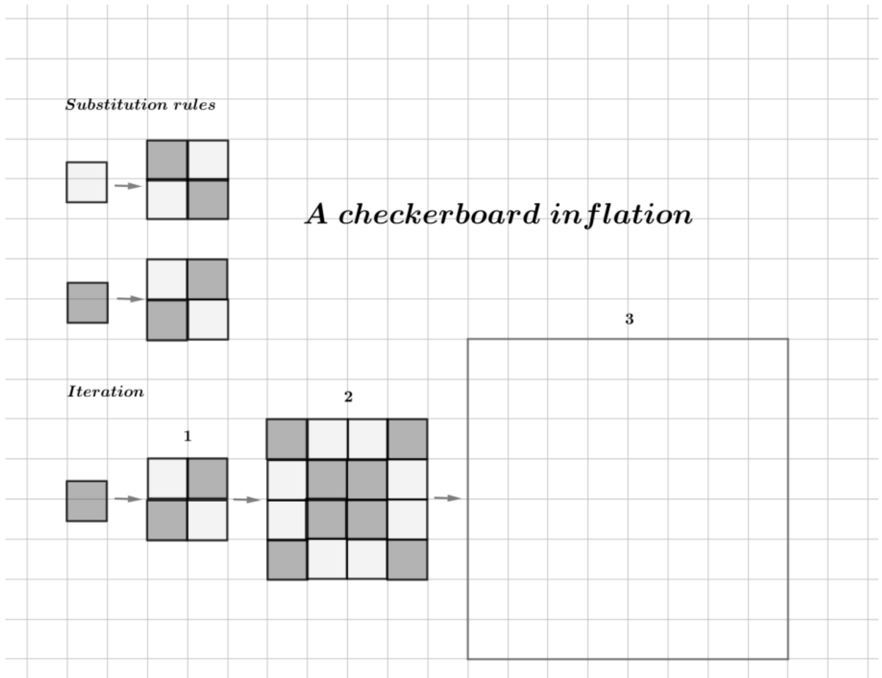
As you see, there are mirror lines here but the pattern is a bit more complicated than you might expect.

E3 Slide 7



This is a famous one. Here, iteration of the substitution rule produces a **fractal**, discovered by Ludwig Koch half a century before we spoke of such things. Using the triangle paper, again, follow the substitution rules to produce the 2nd iteration. Make the first side 9 units long. The interesting thing about this fractal is that, though the area converges on a value 8/5 times the original, the length grows without bound. In other words, the Koch fractal has a finite area but an infinite perimeter.

E4 Slide 8



Draw in the result of the 3^{rd} iteration of this checkerboard example.

Now we come to the main point of the workshop. ***All these patterns grow, but they do not repeat. If we can show that a set of substitution rules leads to an inflation of a tiling, we know that, however big we grow it, it will not repeat by translation, that is to say it will be aperiodic.***

Part 3: The search for sets of tiles forcing aperiodicity

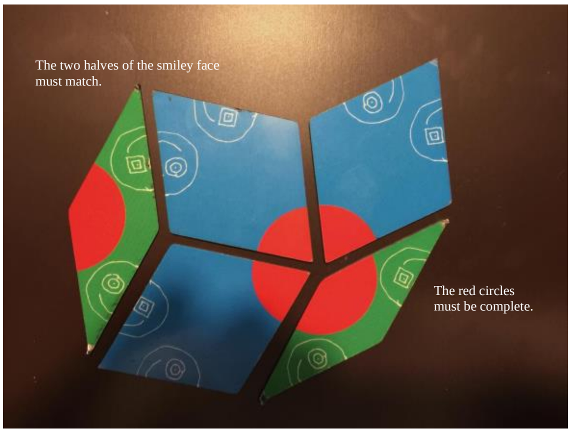
Some history. In the early C20 all the symmetries a periodic pattern could have were known to mathematicians and crystallographers. Around the middle of the century people began to find sets of tiles which would tile the plane only aperiodically. The number of tiles needed in the set went down and down until Roger Penrose, who has recently received the Nobel Prize for his work on black holes, found sets of just two tiles: a kite and a dart, and – the set we shall use – a thick rhombus and a thin one. Remember what a covering is? The mathematician Petra Gummelt found a covering of marked decagons which only covered the plane aperiodically. But the search was still on for a single tile which did so.

Back to the Penrose rhombuses.

Because rhombuses tile periodically, we have to mark the edges in some way.

Slide 9

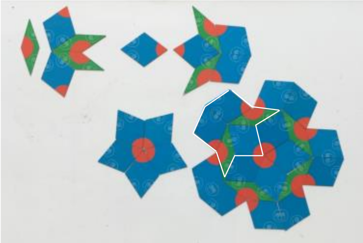
This is how we mark ours.



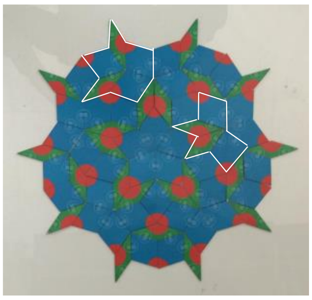
E5 Slide 10

The substitution rules

First inflation of the star



Second inflation of the star



The slide shows the substitution rules for the Penrose tiles. The thin rhomb becomes a bird, perhaps a swallow; the fat rhomb becomes a fish, perhaps an angelfish. It also shows the effect of the first and second iterations on a star chosen to 'seed' the tiling. Notice that the green rhombuses overlap. On the magnet boards, copy part of the result of the second iteration, just to get a feel for how the rhombuses fit together.

We say aperiodic tilings don't repeat by translation, and that's true. The odd thing is that, however big a patch of the tiling you take, as you move across the tiling, you soon meet an exact copy.

Part 4: The discovery of a single such tile

This spring a group of four mathematicians announced that they could tile the plane aperiodically with a single tile, the hat. Single tile = one stone = Einstein in German. So 'the einstein' (small initial letter) = the hat.

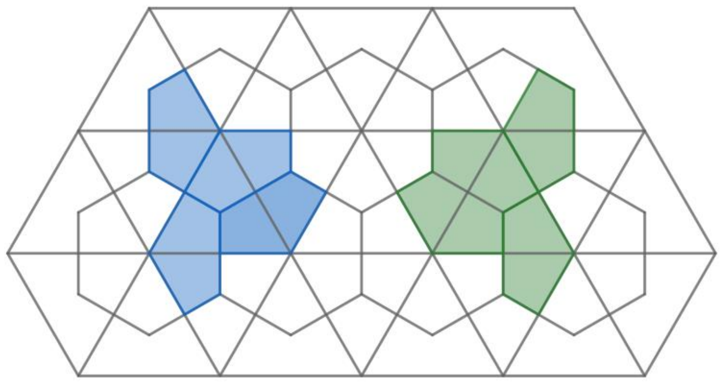
The group consisted of:
David Smith, an amateur mathematician, the initial discoverer,
two mathematicians who had devised computer programs to explore the geometry involved:
Craig S. Kaplan,
Joseph Samuel Myers, and
Chaim Goodman-Strauss, a professional geometer.

The main surprise was that they could draw the hat on a simple grid, the one we looked at before. But Craig Kaplan then found, using a computer program he had devised, that you could keep the angles but change the lengths and you would have an infinite number of tiles that worked.

Animation In this animation I'll point out when three forms appear: the hat, the turtle, (which we won't have time to use today – though it looks much more like a turtle than the hat does a hat!), and the spectre, (which will be important later).

Slide 11

Here is the hat both ways up.



Notice that the kites come in pairs, making an irregular pentagon like a bishop's mitre. How many different interior angles are there and how big are they? [3; $90°, 120°, 240°$]

E6 Fit together some magnetic or plastic hats. When you need one of the opposite handedness, if you're using the magnetic hats, use the second colour; if you're using the plastic hats, turn them over. It turns out that around 1 in 7 have the opposite handedness (**chirality**).

E7 Slide 12

The substitution rules for the hat tiling

U = tile the right way up $\quad$ F = tile flipped over

1. $U \to 7U + F$ (a patch with F in the middle, ringed with U s)
2. $U + F$ (an adjacent tile pair) $\to 6U + F$ (a patch with one less U than the other)

Level 1: U
Iteration 1: $\downarrow$

Level 2: $7U + F$
$6U + (U + F)$
Iteration 2: $\downarrow \quad \downarrow$

$6(7U + F) \quad 6U + F$
Level 3: $48U + 7F$
$41U + 7(U + F)$
Iteration 3: $\downarrow \quad \downarrow$

$41(7U + F) + 7(6U + F)$
Level 4: $329U + 48F$

The substitution rules correspond to a ***recurrence relation***, in which each term is 7 times the last one minus the last-but-one:

a_0	a_1	a_2	a_3	a_4	a_5
0	1	7	48	329	

How do the numbers of U s and F s on each level relate to the numbers in the sequence?

Work out a_5:
1. *using the recurrence relation,*
2. *using the substitution rules to find the number of Us.*

On this slide we follow the first few iterations of the hat tiling. There's a lot on it so let's take the slide apart and put it together again bit by bit. You probably know already one example of a recurrence relation: the one which makes the Fibonacci numbers. There, to get a_2, you just add a_0 and a_1, and carry on like that. $a_3 = a_1 + a_2$, and so on.

Slide 13

On the right is David Smith; on the left, Chaim Goodman-Strauss. The person in the middle is assembling a patch of hats.

Slide 14

Assembling a hat tiling up to level 4 in Newcastle library in September.

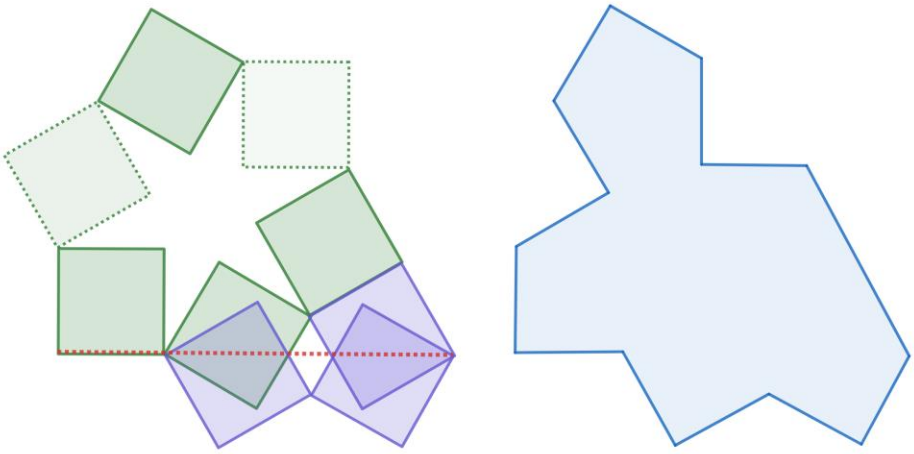
NEWCASTLE LIBRARY 2.9.23

For work of your own, bring up Philip Legner’s site ‘Mathigon’, choose ‘Polypad’, then ‘Aperiodic Tiles’, then ‘Einstein Hat’.

But there was an advance on the hat, which came very shortly after its discovery. As you’ve seen, you have to turn over about 1 in 7 of the hats. Looking back through the infinite family of possible shapes, the team realised that one of the shapes, christened the spectre, didn’t have to be turned over. They had arrived at perfection: a single tile without markings and without having to be turned over which tiles the plane only aperiodically.

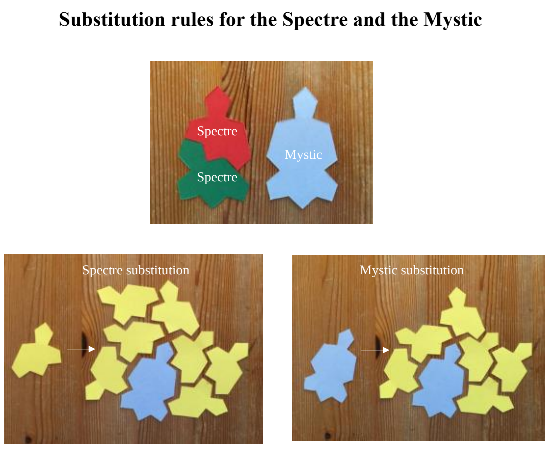
Slide 15

The shape of the spectre.



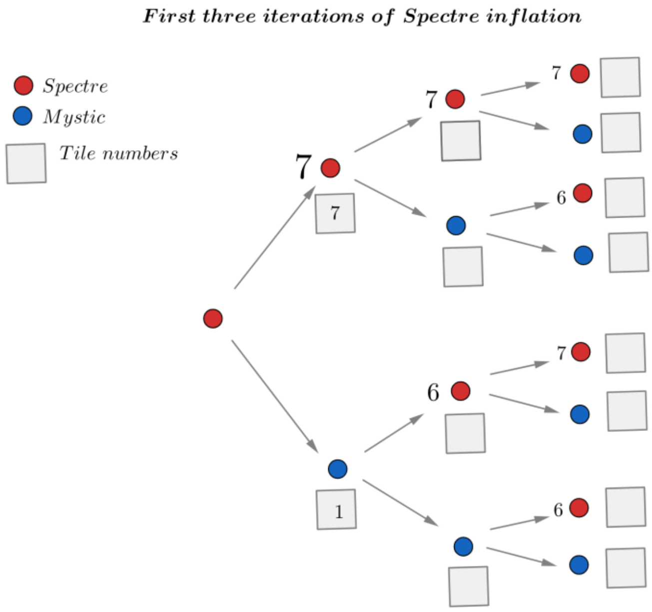
Slide 16

Two spectres making a mystic.
The substitution rules.



E8 Use the magnetic tiles to make a first iteration of the spectre and the mystic.

Slide 17 The first few spectre inflations in terms of the algebra.

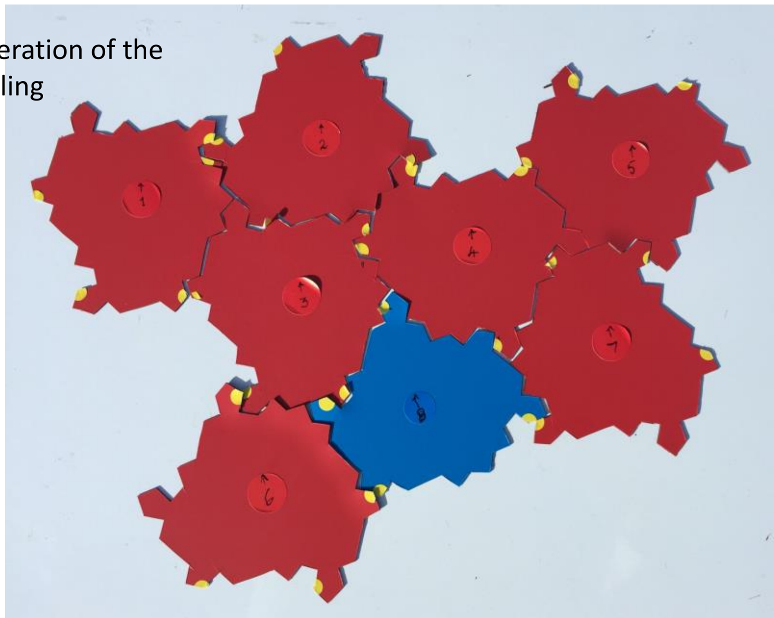


E9 Work out the total number of pieces after each iteration.

Slide 18

Here I’ve made patches from what you’ve just produced and stitched them together to make the second iteration of the spectre tiling.

Second iteration of the Spectre tiling



The last thing to say is that, to the disbelief of crystallographers when they were first described, these aperiodic patterns are found in crystal structures. For his work in the 1980s and on Daniel Schechtman won a Nobel prize in 2011.

Part 5: The lessons to be drawn from this history

What are the lessons to be drawn from this story? [Use the flip chart for suggestions. The list should include the following.]

- *Proofs are the bricks from which the house of mathematics is built.* (We had to know that the hat really would tile the plane without repetition.)
- *The importance of sharing ideas in a group.* (Dave Smith and his colleagues passed ideas backwards and forwards. Google ‘Hedraweb’ and you’ll find the blog he maintains.)
- *The importance of reviewing your results: the ideas may be developed in new directions.* (It was by going back through their findings that they discovered the spectre didn’t need to be turned over.)
- *The possibility over the last century of programming a computer when the number of cases to be examined is too great for work by hand.* (Without Craig Kaplan’s animation it is unlikely the team would have found that the number of single aperiodic tiles is infinite.)

[Write on the flipchart: ‘Mathigon’, ‘Hedraweb’, the source for Craig Kaplan’s animation:cs.waterloo.ca/~csk/hat/animation.mp4]

Draft 8.11.23