FIGURATE NUMBERS

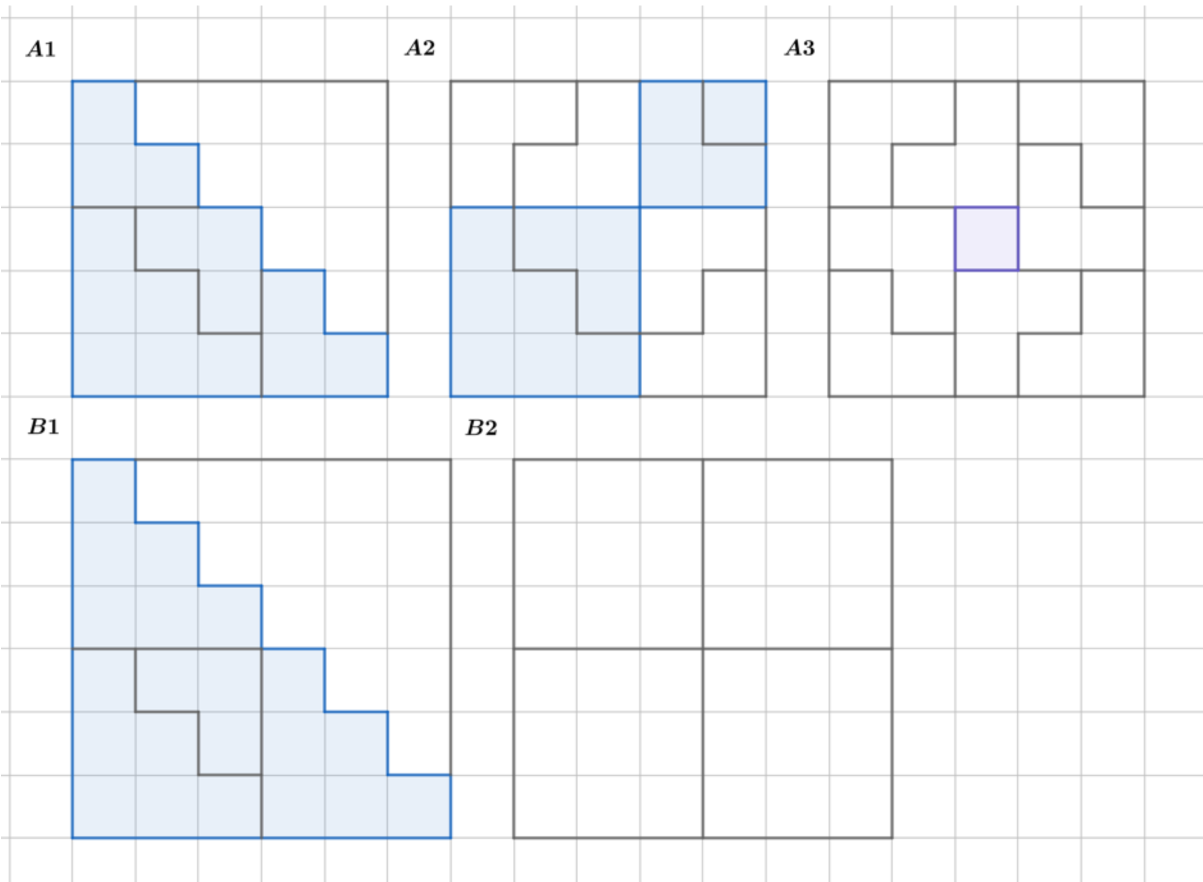
A figurate number is an arrangement of dots, or simple shapes that pack, which represents it. We shall use just eight of the many possible shapes, together with the line of length/column of height n, which we shall simply show as 'n'; and the point, which we shall simply show as '1'.

In accordance with this characterisation, we shall relate numbers using the 'figurate algebra' on the left. We can always convert to common algebra as shown on the right.

T_n will be the n^{th} triangle number,	$\frac{n(n+1)}{2}$
S_n the n^{th} square,	n^2
CS_n the n^{th} centred square number,	$2n(n+1)+1$
CH_n the n^{th} centred hexagon number,	$3n(n-1)+1$
Tet_n the n^{th} tetrahedral number,	$\frac{n(n+1)(n+2)}{6}$
Pyr_n the n^{th} pyramidal number,	$\frac{n(n+1)(2n+1)}{6}$
Oct_n the n^{th} octahedral number	$\frac{n(2n^2+1)}{3}$
C_n the n^{th} cube.	n^3

Triangles and squares

The figure shows different ways to dissect odd squares, A, and even squares, B .



A1, *B1* show the basic relation $T_{n-1} + T_n = S_n$. **(1)**

A2 shows $S_{2n+1} = S_n + S_{n+1} + 4T_n$. We can break this down:

$$\begin{aligned} S_{2n+1} &= (T_{n-1} + T_n) + (T_n + T_{n+1}) + 4T_n \\ &= (T_{n-1} + T_{n+1}) + 6T_n \\ &= [T_n - n] + [T_n + (n+1)] + 6T_n \\ &= 2T_n + 1 + 6T_n = 8T_n + 1, \text{ which is shown by } A3. \end{aligned}$$ **(2)**

Note the identity derived in red: $T_{n-1} + T_{n+1} = 2T_n + 1$. **(3)**

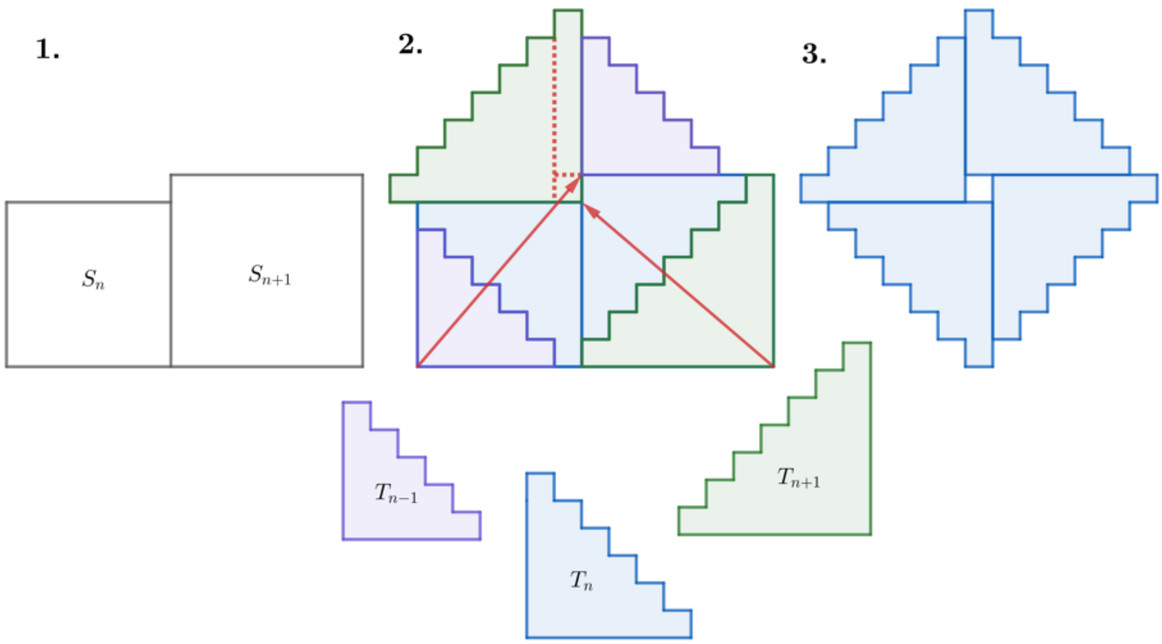
Note also the rotational symmetry by which 4 rectangles, each comprising 2 T_n triangles, are arranged round the central square.

A1 also shows $T_{2n+1} = 3T_n + T_{n+1}$. **(4)**
The corresponding relation in *B1* is $T_{2n} = T_{n-1} + 3T_n$. **(5)**

B2 simply shows the relation: $S_{2n} = S_2 S_n$, which generalises to: $S_{tn} = S_t S_n$. **(6)**

Many other identities can be derived, either from the algebra or by dissecting the figures in different ways.

Another identity arises from the sum of consecutive squares:



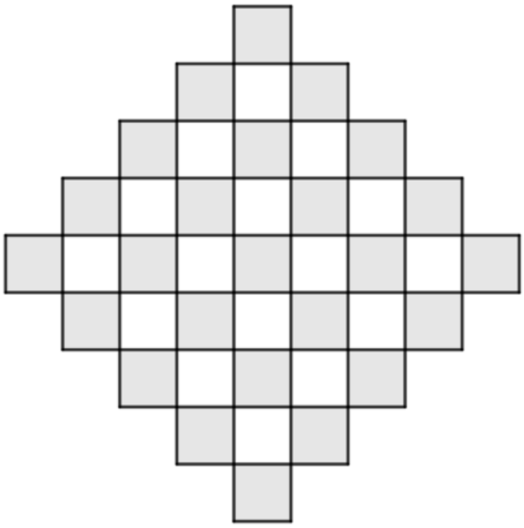
1. We begin with our squares.
2. We divide each into consecutive triangles, so we have:

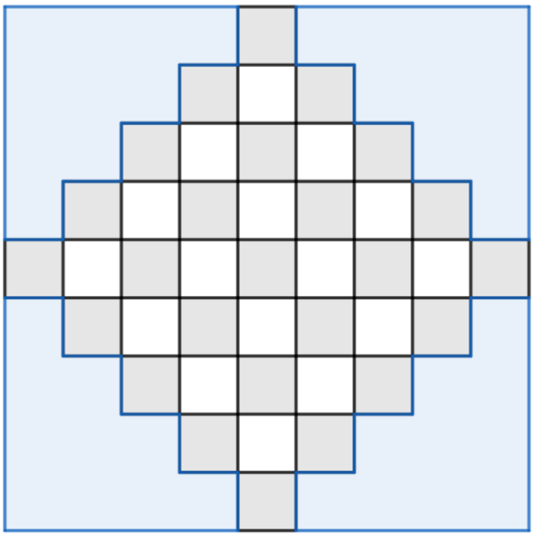
$S_n + S_{n+1} = T_{n-1} + T_{n+1} + 2T_n$. **(7)**

We slide the green triangle bottom right to top left, and the lilac triangle bottom left to top right. We then make the cuts shown by the dotted lines. This does geometrically what we represented algebraically in identity **(3)**.

3. The net result is the 'centred square' number, $CS_n = S_n + S_{n+1} = 4T_n + 1$. **(8)**

The next figure shows part of a tiled floor with a checkerboard pattern. If you turn your head 45° to the vertical, the two consecutive squares reveal themselves. Alongside, we see what happens when we subtract **(8)** from **(2)**: $S_{2n+1} - CS_n = 4T_n$. **(9)**





Tetrahedra, Pyramids, Octahedra

We now move into the third dimension.

To make a tetrahedron, we stack triangles. To make a pyramid, we stack squares.
We can compare the figures layer by layer.

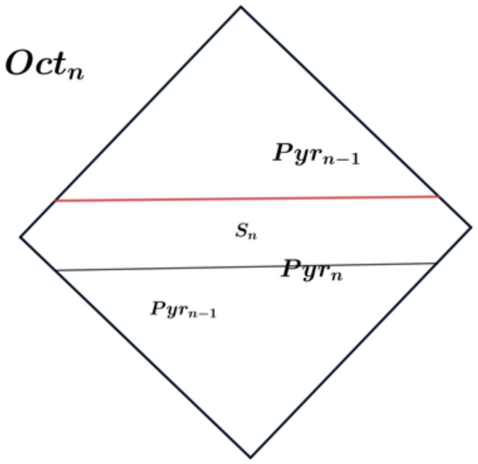
As we had $T_{n-1} + T_n = S_n$, we have $Tet_{n-1} + Tet_n = Pyr_n$. **(10)**

Here is another identity.

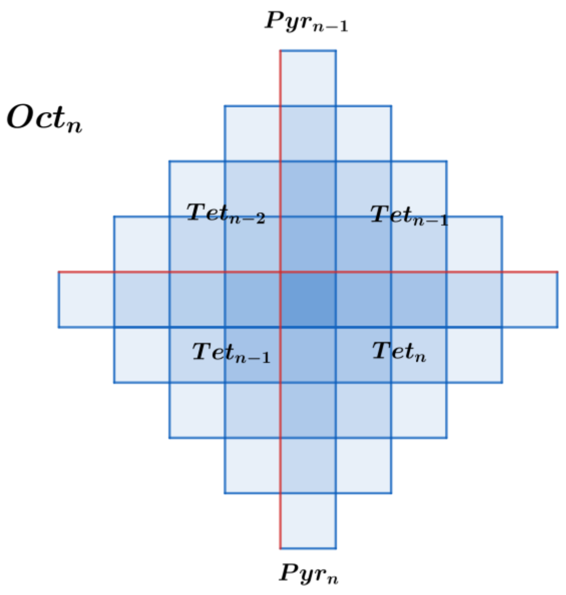
$$\begin{aligned} Tet_{2n} &= (T_1 + T_2) + (T_3 + T_4) + (T_5 + T_6) + \cdots + (T_{2n-1} + T_{2n}) \\ &= S_2 + S_4 + S_6 + \cdots + S_{2n} \\ &= 4S_1 + 4S_2 + 4S_3 + \cdots + 4S_n \\ &= 4(S_1 + S_2 + S_3 + \cdots + T_n) \\ &= 4Pyr_n. \end{aligned}$$ **(11)**

And the pattern $X_{n-1} + X_n = Y_n$ continues. To make an octahedron, we take a pyramid, and stick the base to an inverted pyramid one layer bigger. $Pyr_{n-1} + Pyr_n = Oct_n$. **(12)**

Here is a schematic section:



Here is the octahedron represented as a centred square pyramid and seen in plan:

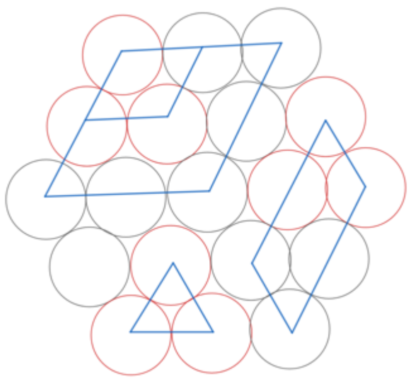


A centred square is a sum of consecutive squares. Each layer on the right combines one layer from the upright pyramid with one from the inverted one.

If we isolate the central column, as we did the square in two dimensions, we have the three-dimensional analogue of **(8)**:

$$Oct_n = 4Tet_{n-1} + n. \quad \textbf{(13)}$$

If we use circles instead of squares, we can arrange 6 circles round a central circle to make the centred hexagon number CH_n:



We can dissect the figure in several ways. From the triangles we have:

$$CH_n = 6T_{n-1} + 1. \quad \textbf{(14)}$$

Interesting are the larger rhombuses. If we divide the hexagon into three rhombuses, swap the circles for spheres, and fold so that the lines of spheres at 120° become mutually perpendicular, we shall have the gnomon to a cube. That is to say,

$$C_{n-1} + CH_n = C_n. \quad \textbf{(15)}$$

Regular tetrahedra pack with regular octahedra to fill space. Likewise tetrahedra and octahedra made from packed spheres. This gives us the identity

$Tet_{2n+1} = 4Tet_n + Oct_{n+1}$. **(16)**

From **(13)** we have $Oct_{n+1} = 4Tet_n + (n+1)$. Substituting in **(16)**,
$Tet_{2n+1} = 8Tet_n + (n+1)$. **(17)**