Figurate Identities

The recreational algebra of number shapes

Aspects of elementary number theory visualised

A sourcebook of images for
junior and high school teachers

Contents

Introduction

explaining what the book's about
and how it's laid out

Definition

A figurate number is a shape consisting of an array of units which pack (circles, squares, hexagons, spheres, cubes) representing that particular class of number. For example, an equilateral triangle dissected out of a close packing of circles or a right isosceles triangle dissected out of a square grid denotes the general triangle number T_n representing the expression $\frac{n(n+1)}{2}$.

The choice of shapes

We use a dozen and a half of the many possible shapes. Our choice follows convention except in five cases: the twin trapezoid, the alternate hexagon, the Greek gnomon, the cuboctahedron and the icosahedron shell. The n^{th} s-sided polygon number, $P_{s,n}$ can be dissected as $(s-3)T_{n-1}+T_n$. We have used $s=3$, the triangle, T_n, and $s=4$, the square, S_n, but not $s=6$, the hexagon number, H_n. First, by virtue of the identity $3T_{n-1}+T_n=T_{2n-1}$, it can be included with the triangle numbers. Second, $H_{n-1}+H_n \neq CH_n$, so we can make no analogy with the centred squares and cubes.

The shapes are of two kinds: single-parameter, e.g. T_n, and two-parameter, e.g. $Tr_{m,n}$. The latter include the former as a limiting case. In that example, a triangle is a trapezoid where one of the parallel sides vanishes.

Notation

We shall relate numbers using the 'figurate algebra' on the left below. We can always convert to the common algebra on the right.

Figurate algebra	**Conventional algebra**
1, the unit	1
L_n , the n^{th} line number	n
T_n , the n^{th} triangle	$\frac{n(n+1)}{2}$
S_n the n^{th} square	n^2
O_n, the n^{th} odd number	$2n-1$
$GG_{m,n}$, a Greek gnomon	$(m-n)(m+n)$
$Tr_{m,n}$, a trapezoid	$\frac{(m-n)(m+n+1)}{2}$
$DTr_{m,n}$, a twin trapezoid	$(m-n)(m+n)-n$
CS_n, the n^{th} centred square	$2n(n-1)+1$
CH_n, the n^{th} centred hexagon	$3n(n-1)+1$
Tet_n, the n^{th} tetrahedron	$\frac{n(n+1)(n+2)}{6}$
Pyr_n, the n^{th} pyramid	$\frac{n(n+1)(2n+1)}{6}$
Oct_n, the n^{th} octahedron	$\frac{n(2n^2+1)}{3}$
C_n, the n^{th} cube	n^3
CO_n, the n^{th} cuboctahedron	$\frac{10n^3-15n^2+11n-3}{3}$
CC_n, the n^{th} centred cube	$(2n-1)(n^2-n+1)$
I_n, the n^{th} icosahedron shell	$10n^2+2$

We take $n \geq 1$ (or some other value according to context). For those shapes with central symmetry, the suffix 1 denotes the unit, thus $S_1 = 1$, etc., as is conventional. Unconventional is naming *n* a 'line number' and giving it the symbol L_n, which we do only for consistency.

We use conventional algebra where the figurate form would be clumsy, e.g. 'T_{2n+1}', not '$T_{O_{n+1}}$', 'T_{2n}', not 'T_{2L_n}'. Sometimes for concision we write 'triangle' for 'triangle number', etc.

Housekeeping

We observe *dimensional consistency*. When we write '$C_n - S_n$', we understand that the 'S_n' is a square prism of unit height. When we write '$Tet_{n-1} + T_n = Tet_n$', we understand that, again, the 'T_n' is a prism of unit height. An interesting example we shall meet is $\sum_{i=1}^{i=n} i^3 = (T_n)^2$. The sequence of cubes on the left has dimension 3 + 1 = 4. This convention accords with the rule that, with each descending diagonal on Pascal's Triangle and those derived from it, the dimension increases by 1. The square of the triangle on the right has dimension 2^2 = 4 also. It may be necessary to expand expressions to determine their dimension. For example, in **[3.7]** we meet T_{S_n}, that is,
$\frac{S_n S_{n+1}}{2}$, which has dimension 4.
In **[3.8]** we meet $(T_n)^4 - (T_{n-1})^4$, that is,
$[(T_n)^2 + (T_{n-1})^2](T_n + T_{n-1})(T_n - T_{n-1}) = [(T_n)^2 + (T_{n-1})^2]S_n L_n$,
which has dimension 7.

We are concerned throughout with the positive integers, the natural numbers. Hence all our equations are Diophantine. When we write '$(a - b)$' we assume a > b. We do not use double implication arrows when simply manipulating algebraic expressions. When that manipulation is routine, we leave it to the student to complete.

Method

We derive the identities of figurate algebra by examining shapes which result from combining smaller ones. The ideal procedure is:

(A) Examine the shape → **(B)** Derive a figurate identity → **(C)** Check by translating into common algebra → **(D)** Use common algebra to see if the identity generalises further → **(E)** If it does, try to represent the generalisation graphically.

In only a few cases do we advance through all five stages.

The fundamental relations

Relations between figurate numbers can be of several kinds and involve one or more classes but the most productive are of two sorts:

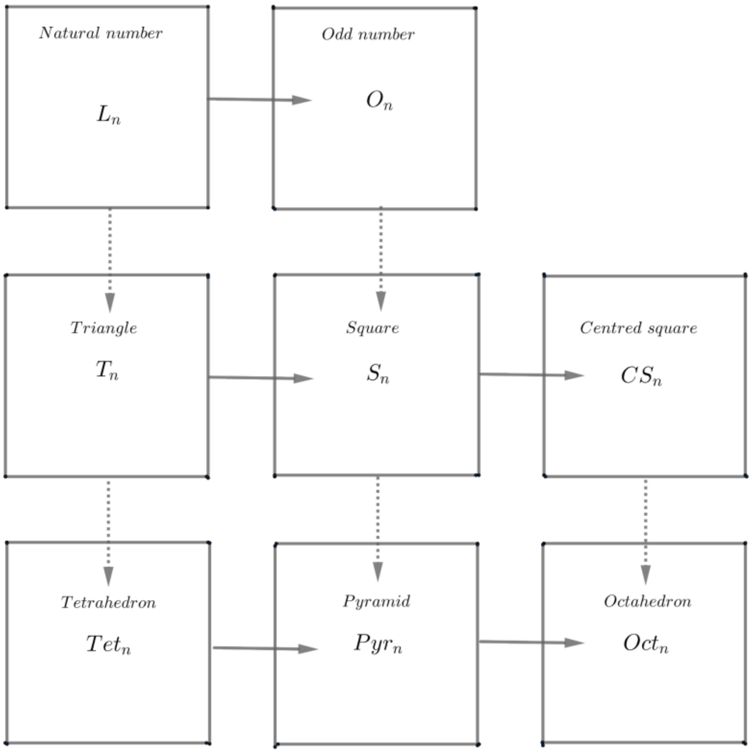
(**a**) $P_{n-1} + P_n = Q_n$. Example: $T_{n-1} + T_n = S_n$.

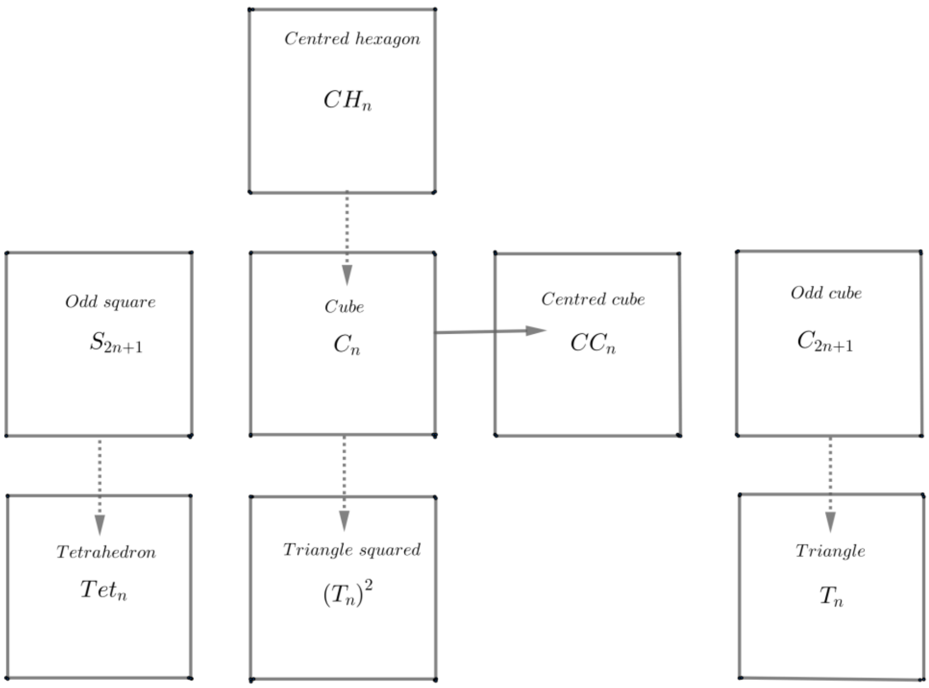
Two consecutive terms in a sequence generate a different shape of the same dimension.

(b) $P_{n-1} + G_n = P_n$, or $\sum_{i=1}^{i=n} G_i = P_n$. Example: $S_{n-1} + O_n = S_n$.

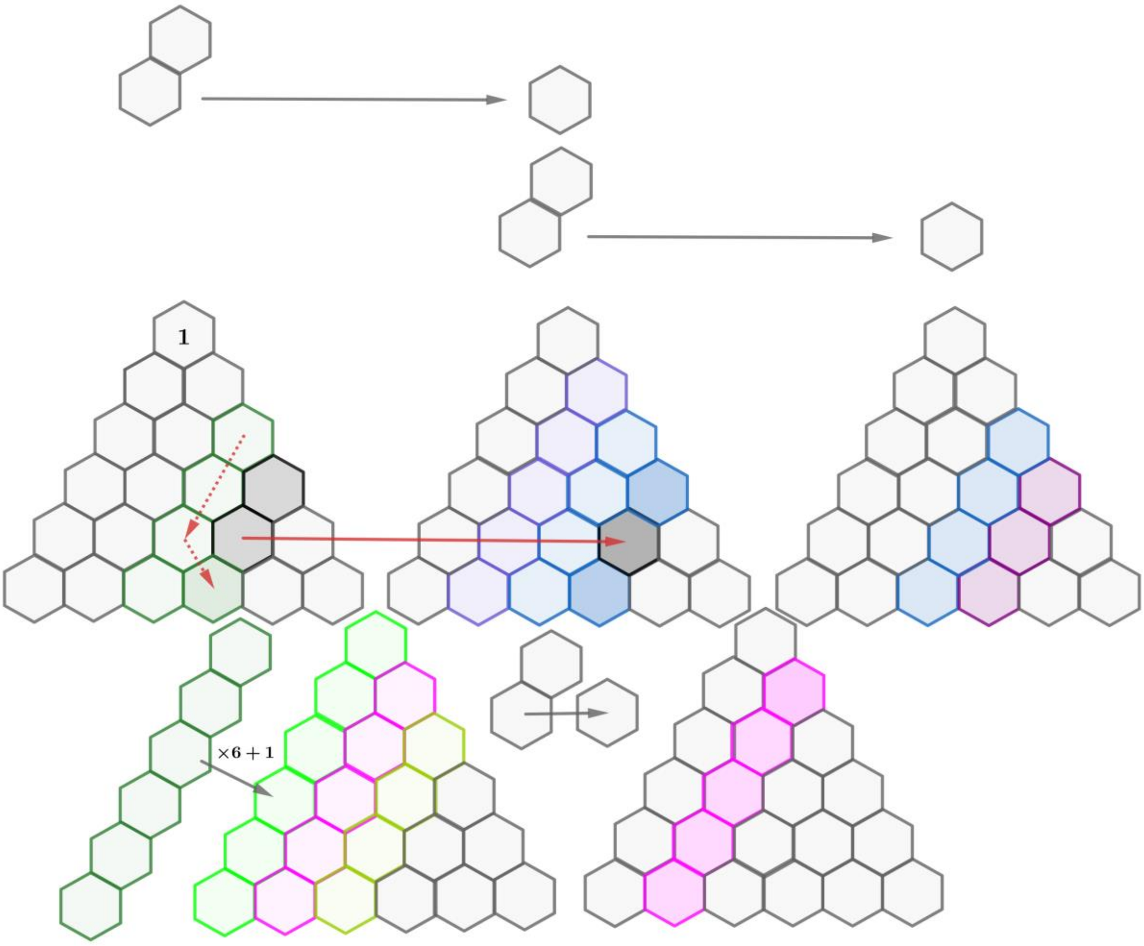
A figure which, added to an existing figure, completes a similar one, is called a *gnomon*.
The 'G' figure is gnomon to the 'P' figure. The gnomon is one dimension less than the figure.
(In the text, amongst other types defined, I distinguish this as the *parent* gnomon.)

In both **(a)** and **(b)** we add a pair of terms to complete the shape; in **(b)**, this addition can be iterated.
The gnomonic relation, **(b)**, allows students to prove by induction the results we derive.
Showing **(a)** as a solid horizontal arrow and **(b)** as a dotted vertical arrow, we have these relational grids.
(The suffices here just label the general form.)





We show the relations above on Pascal's Triangle and number arrays of the Pascal type derived from it. A solid arrow represents a transformation of one array into another by the following rule: two cells in the position shown at the tail of the arrow sum to one at the head of the arrow. In the example below, two tetrahedron numbers sum to a pyramid number. A dotted arrow represents the descent of a diagonal, whose sum appears in a cell below and to one side by virtue of the 'Christmas stocking' theorem. In the example, the first three triangle numbers sum to the third tetrahedron number.



Where applicable, we use this figure to head each chapter, shading in the diagonals which contain numbers of the types to be discussed and labelling them with the correct symbols.

On occasion, we show how the numbers appear on the operation table for multiplication.

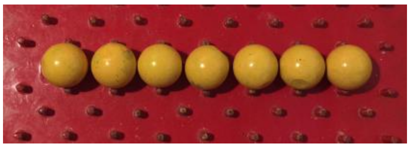
In the course of our survey we meet some classic 'proofs without words'. Perhaps these will encourage the reader to seek his or her own.

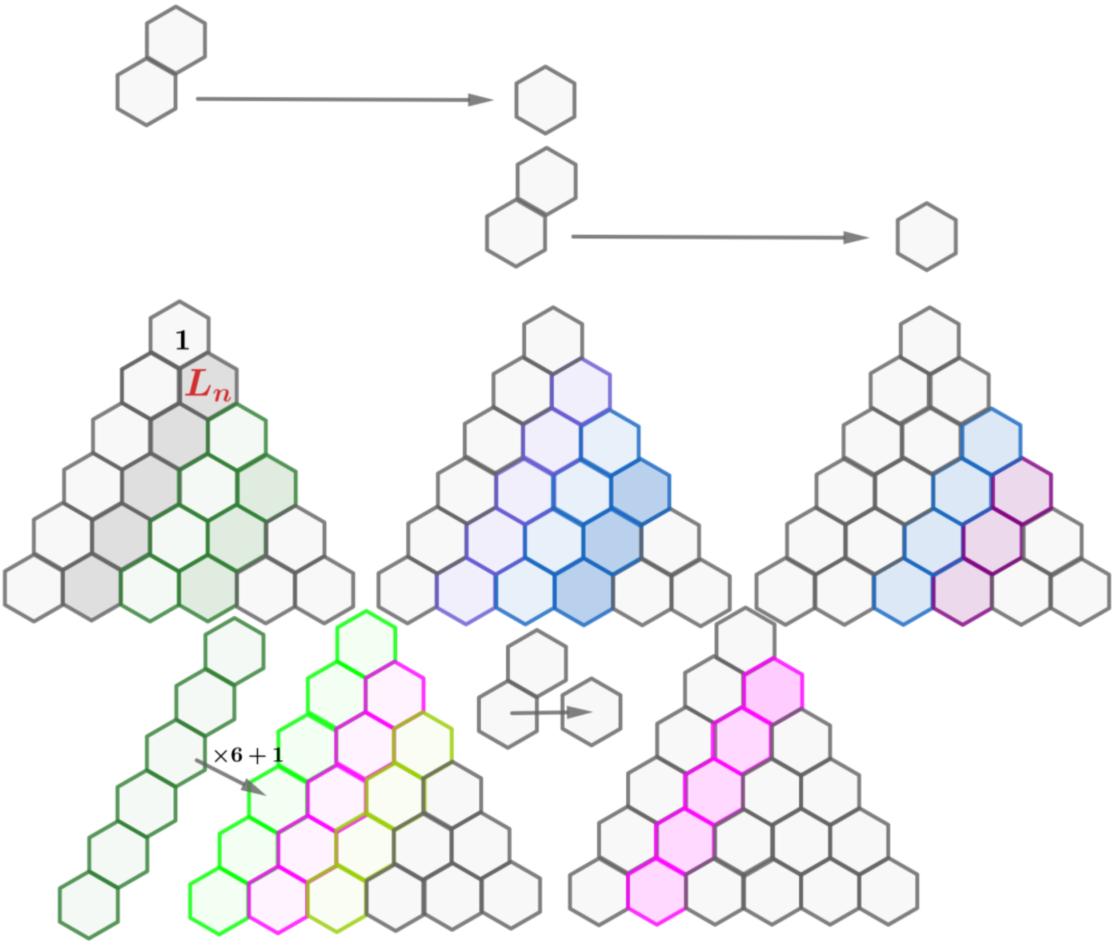
The identities listed in this book fall in the section 'Integer sums' in the three books edited by Roger B. Nelsen: 'Proofs without Words', 'Proofs without Words II', 'Proofs without Words III'.

Though our main topic is the identities themselves, we also describe numerical properties where those illuminate a relationship.

Chapter 1

The Line, L_n





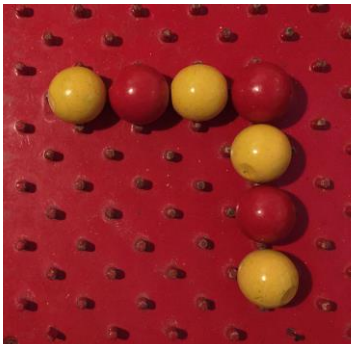
The *line*, L_n, is a strip of length *n* and unit width. No addition of units creates a shape similar to the original, so we cannot find a gnomon to the line. We shall say instead that the unit *completes* the line, preserving an identity of the gnomonic type:

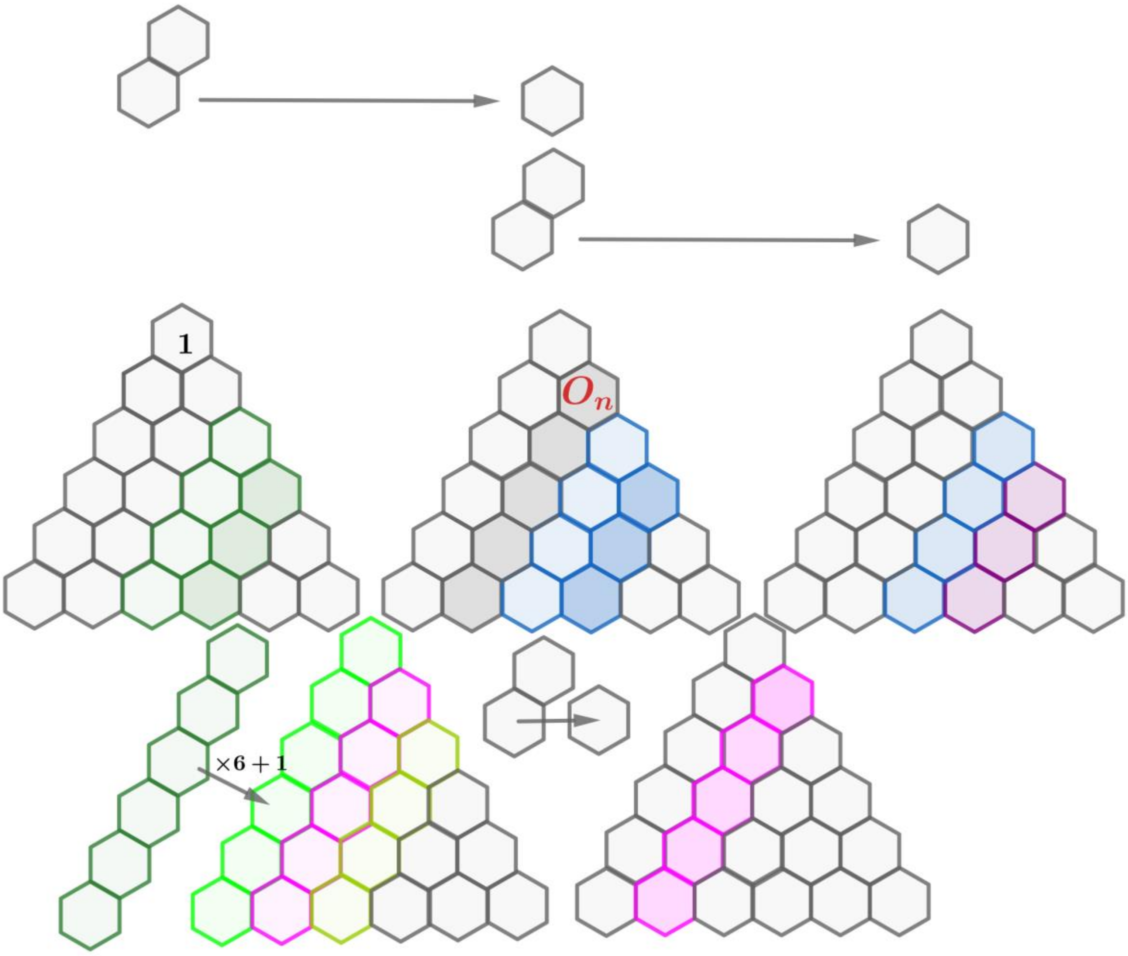
$$\boldsymbol{L_{n-1} + 1 = L_n.} \qquad \textbf{[1.1]}$$

Line numbers account for all positive integers, including necessarily all primes.

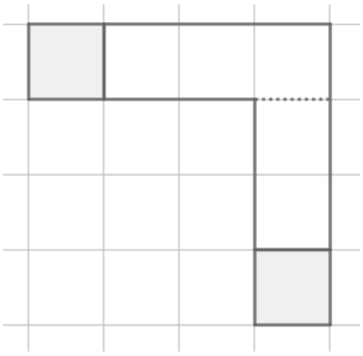
Chapter 2

The odd Number, O_n





We represent the odd number, $O_n = 2n - 1$, as a symmetrical L-shape:



This time 2 units complete the figure as shown, and we can write

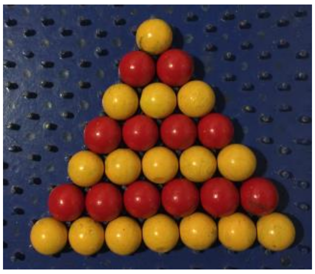
$$\boldsymbol{O_{n-1} + 2 = O_n}. \qquad \textbf{[2.1\}}$$

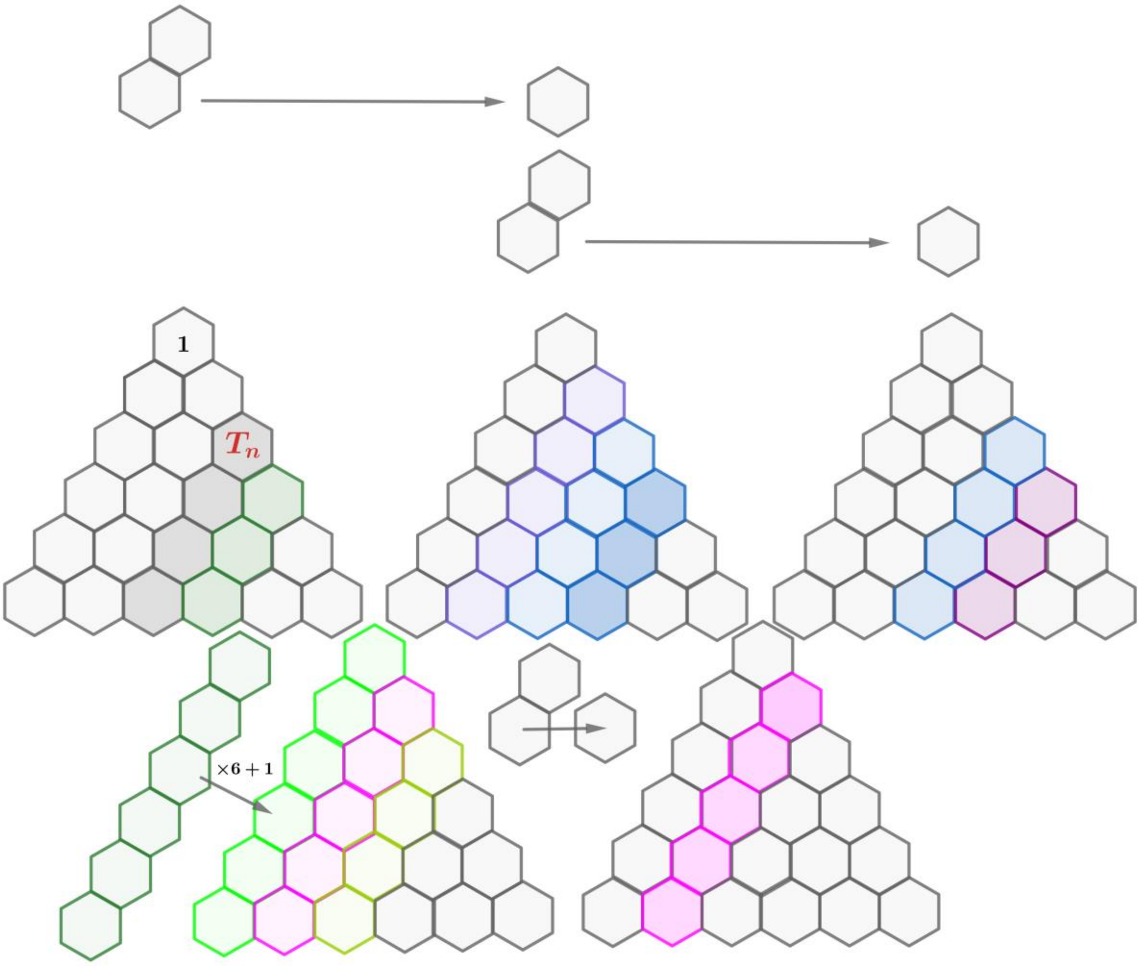
By virtue of the partition indicated by the dotted line, we can also write

$$\boldsymbol{L_{n-1} + L_n = O_n}. \qquad \textbf{[2.2]}$$

Chapter 3

The Triangle, T_n

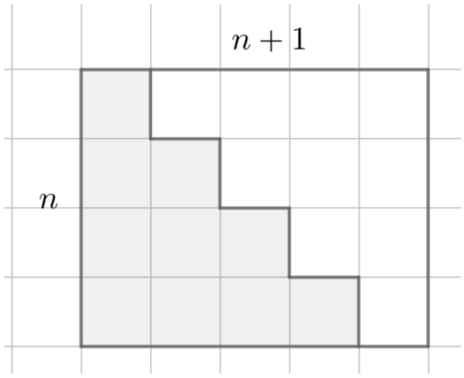




(A) The triangle and shapes derived from it

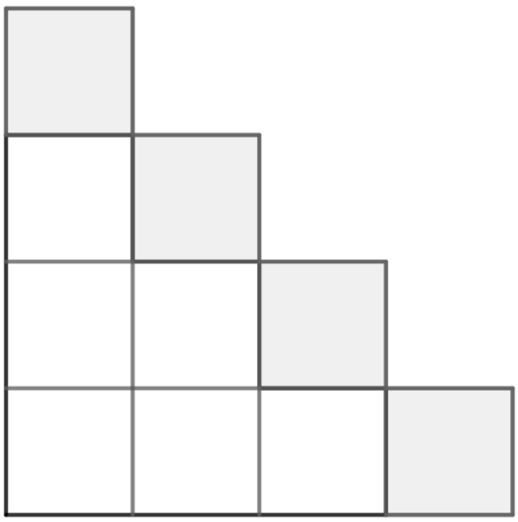
(a) The triangle

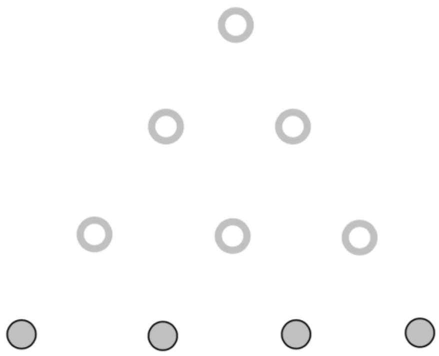
The formula for T_n, $\frac{n(n+1)}{2}$, is clear from this canonical figure:



As the following, equivalent, representations show, we can characterise L_n as gnomon to a triangle, giving the identity

$$\boldsymbol{T_{n-1} + L_n = T_n}. \quad \textbf{[3.1]}$$

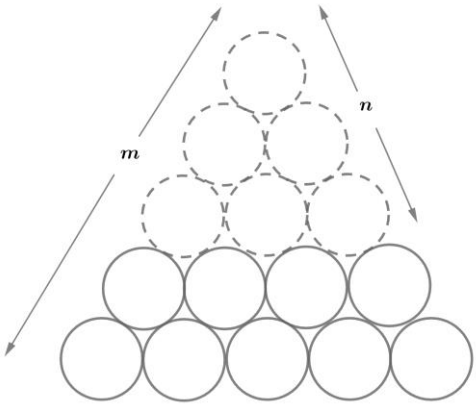




[3.1] serves for a whole series of identities we meet in the text of the form $T_{X-1} + X = T_X$.

(b) Trapezoids, $\boldsymbol{Tr_{m,n}}$

The trapezoid $Tr_{m,n}$ is a sum of consecutive integers. As such it can be characterised as the difference between the triangle number T_m and the triangle number T_n, $m > n$, $n \geq 0$.

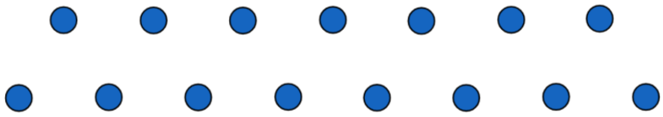


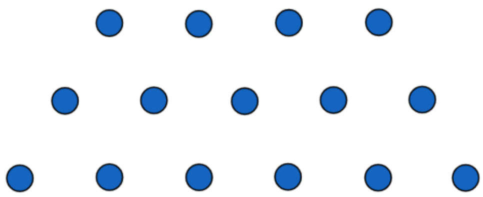
We have:

$$T_m - T_n = \frac{m(m+1)-n(n+1)}{2} = \frac{(m-n)(m+n+1)}{2}\,.$$

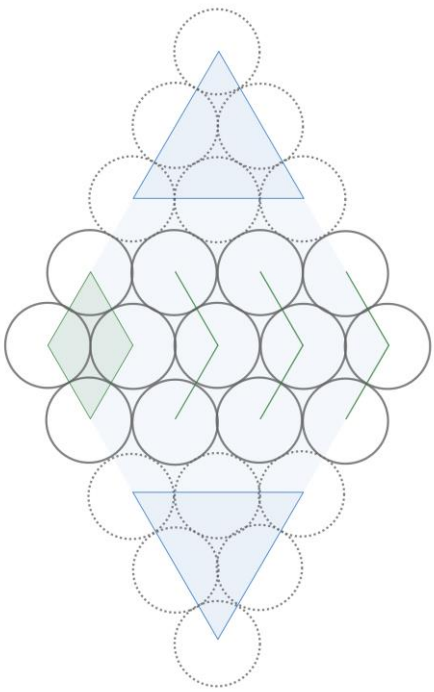
The brackets are of opposite parity. When $n = 0$ we have the triangle. Since $m \geq 1$, there will always be an odd factor, so, excepting $2^0 = 1$, no triangle number can be a power of 2.
More generally, this is true of all $T_{m,n}$ when $m > 1$ (as it is often defined).

In the final part of this chapter we find triangle numbers which are differences between different pairs of triangles. For example, $T_5 = T_8 - T_6 = Tr_{8,6}$ and also $T_6 - T_3 = Tr_{6,3}$. Therefore trapezoids, which include all triangles, do not necessarily represent a number in a unique way. Here are the last two figures:





(c) Twin trapezoids, $TTr_{m,n}$



The figure shows a twin trapezoid. It is a trapezoid combined with its reflection in the base line.

(i) We can characterise it as shown in blue, as a square from which two equal triangles have been removed.

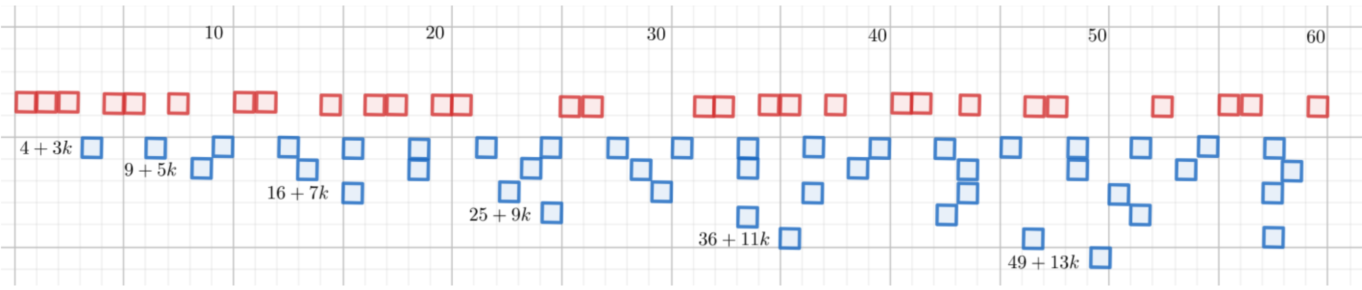
(ii) Alternatively, we can characterise it as shown in green, as the square S_{m-n}, to which *n* copies of O_{m-n} have been added:

$TTr_{m,n}$
$= S_m - 2T_n = S_{m-n} + nO_{m-n}$
$= (m+n)(m-n) - n$.

We require $m - n > 1$, $n \geq 0$. When $n = 0$, we have a square.

Using characterisation (ii), we see that the values are given by $t^2 + k(2t-1)$ as *t* ranges over all integers > 1 and *k* ranges over all integers > 0.

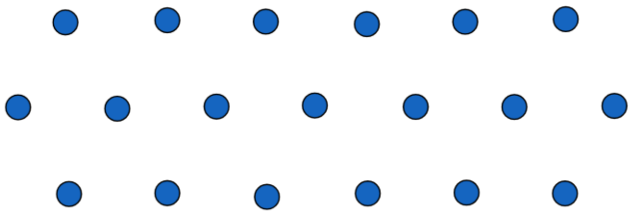
On the following chart red squares denote numbers which are not twin trapezoids. Out of the first 60 natural numbers, 31, just over half, are twin trapezoids.

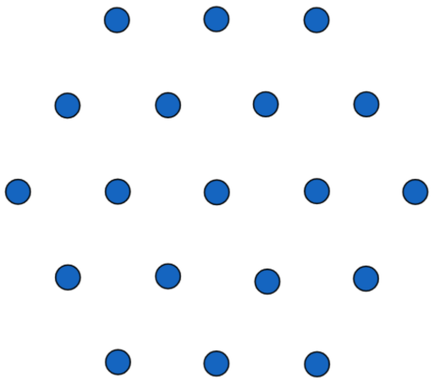


If we dissect the twin trapezoid into two trapezoids, one of which contains the centre line, we have the type **(a)** relation

$$\boldsymbol{Tr_{m-1,n} + Tr_{m,n} = TTr_{m,n}} \quad \textbf{[3.2]}$$

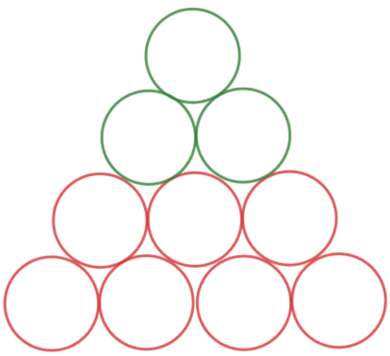
As there may be different representations of a number as a trapezoid, there may be alternative representations as a twin trapezoid. Twin trapezoids include not only all squares, the subject of **Chapter 2**, but also all centred hexagon numbers, the subject of **Chapter 6**. Here is 19 shown in two ways, the second of which is a centred hexagon:

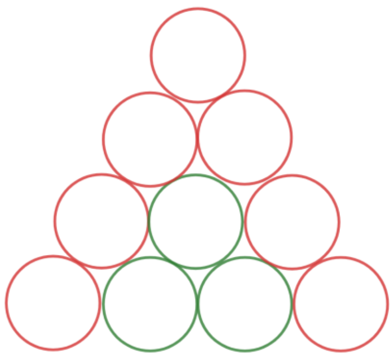




(d) The gnomon and its relatives

We can consider that a gnomon as defined above completes a parent figure.
Call a gnomon which completes a *second* generation figure a *grandparent*.
Call a gnomon which completes a *third* generation figure a *great-grandparent.*
For the triangle, we can show a grandparent in two ways:

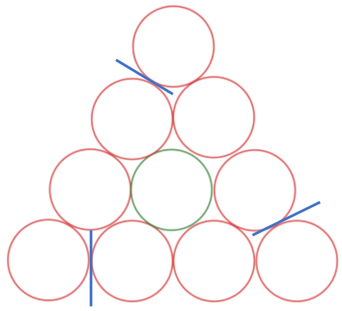




Taking advantage of **[2.2]**, we see that the resulting identity is

$$T_{n-2} + (L_{n-1} + L_n) = \boldsymbol{T_{n-2} + O_n = T_n}. \textbf{ [3.3]}$$

We can draw the great-grandparent completely enclosing the figure:



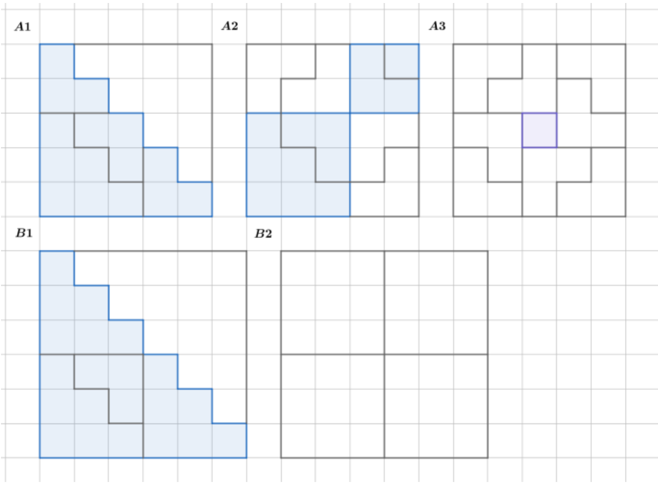
We have:

$$T_{n-3} + (L_{n-2} + L_{n-1} + L_n) =$$
$$\boldsymbol{T_{n-3} + 3L_{n-1} = T_n}\ . \qquad \textbf{[3.4]}$$

The coefficient '3' is matched by the order of rotation symmetry of the figure.

(e) Identities involving $\boldsymbol{T_n}$

The following figure shows different ways to dissect odd squares, the *A* series; and even squares, the *B* series.



A1, *B1* show the relation $\boldsymbol{T_{n-1} + T_n = S_n}$. **[3.5]**

On this multiplication square we show where the triangle numbers appear and how they sum to squares:

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

A2 shows $S_{2n+1} = S_n + S_{n+1} + 4T_n$. We can use **[3.1]** and **[4.3]** to break this down:

$$\begin{aligned} S_{2n+1} &= (T_{n-1} + T_n) + (T_n + T_{n+1}) + 4T_n \\ &= (T_{n-1} + T_{n+1}) + 6T_n \\ &= [T_n - L_n] + [T_n + L_{n+1}] + 6T_n \\ &= 2T_n + 1 + 6T_n = 8T_n + 1, \end{aligned}$$

$\boldsymbol{S_{2n+1} = 8T_n + 1}$.**[3.6]**

which is shown immediately by *A3*.

Note the identity derived in red: $\boldsymbol{T_{n-1} + T_{n+1} = 2T_n + 1}$.**[3.7]**

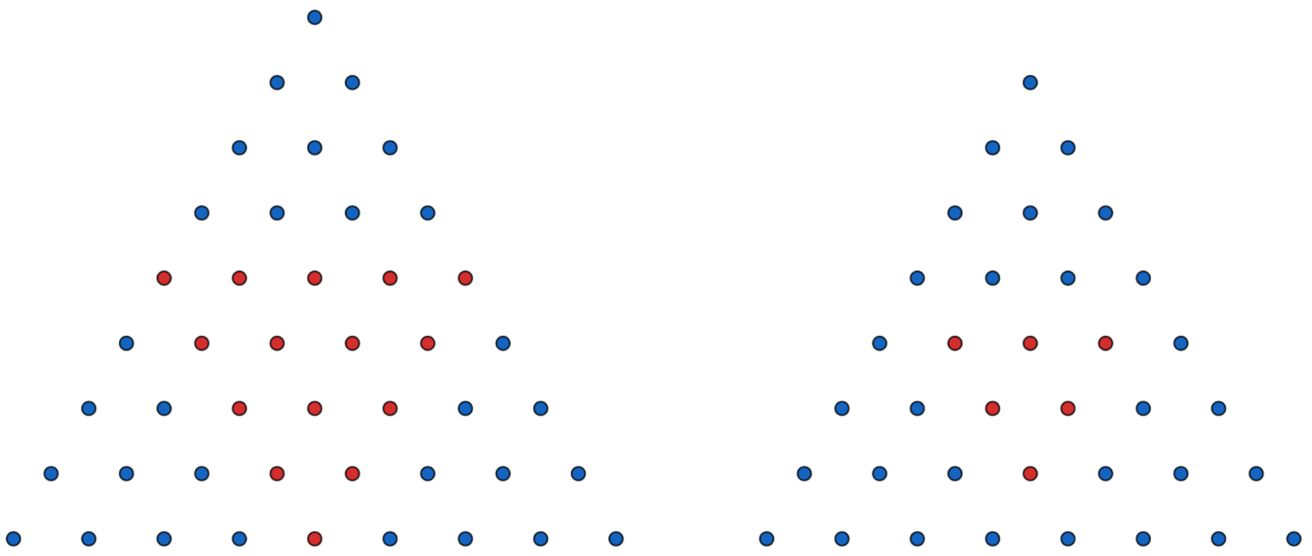
A1 also shows $\boldsymbol{T_{2n+1} = 3T_n + T_{n+1}}$. **[3.8]**

The corresponding relation in *B1* is $\boldsymbol{T_{2n} = T_{n-1} + 3T_n}$. **[3.9]**

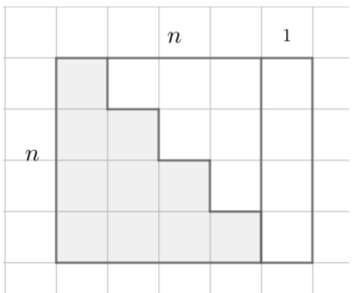
B2 shows the relation: $S_{2n} = S_2 S_n$, which generalises to:

$\boldsymbol{S_{tn} = S_t S_n}$. **[3.10]**

Here are alternative figures for **[3.8]**, **[3.9]** respectively, emphasising symmetry:



Redrawing our canonical figure like this:

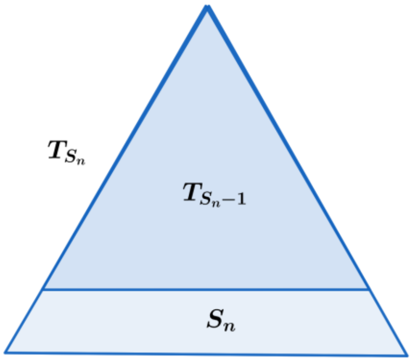


we have the identity $\boldsymbol{L_n + S_n = 2T_n}$. **[3.11]**

We can derive it algebraically like this:

$L_n + S_n$
$= L_n + (T_{n-1} + T_n)$
$= (L_n + T_{n-1}) + T_n$
$= T_n + T_n$
$= 2T_n$. We note that $2T_n = n(n+1)$ has been known, following Aristotle, as a *pronic* number. (Nelsen calls it an *oblong* number.)

The following diagram illustrates the identity $\boldsymbol{T_{S_n} - T_{S_n - 1} = S_n}$. **[3.12]**



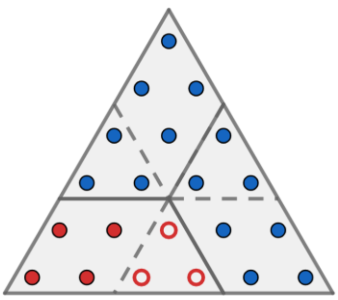
We can use either **[3.6]**, **[3.7]**, or **[3.8]**, **[3.9]** to derive $\boldsymbol{T_{n-1} + 6T_n + T_{n+1} = S_{2n+1}}$. **[3.13]**

In the following figures we dissect the triangle in a number of different ways, always observing rotation symmetry. In the first row we use trapezoids. Note how we triple-count the overlap region in the middle case. In the second row we use line numbers but we have notated them as trapezoids one row thick.

$3Tr_{2n,n}$ [3.14]
$3(T_{2n} - T_n)$ [3.15]

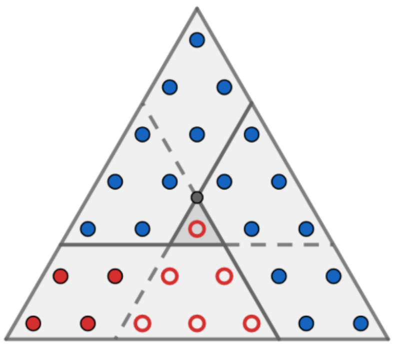
$3Tr_{2n+1,n+1} + 1$ [3.20]
$3(T_{2n+1} - T_{n+1}) + 1$ [3.21]

$3Tr_{2n+1,n}$ [3.26]
$3(T_{2n+1} - T_n)$ [3.27]



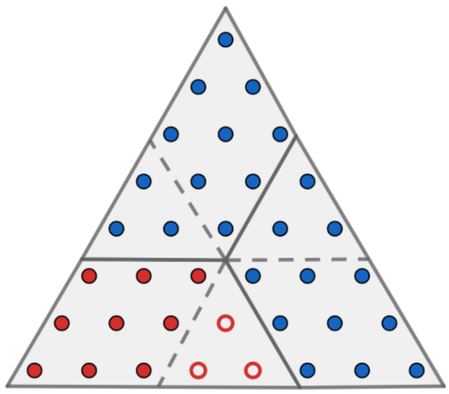
T_{3n}

$3(S_n + T_n)$ [3.16]
$3(T_{n-1} + 2T_n)$ [3.17]



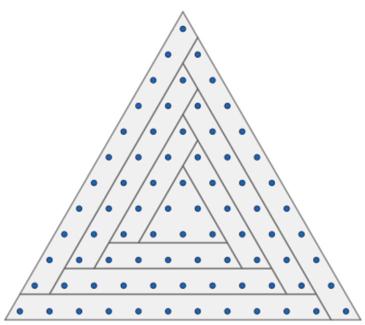
T_{3n+1}

$3(S_n + T_{n+1}) - 2$ [3.22]
$3(T_{n-1} + T_n + T_{n+1}) - 2$ [3.23]

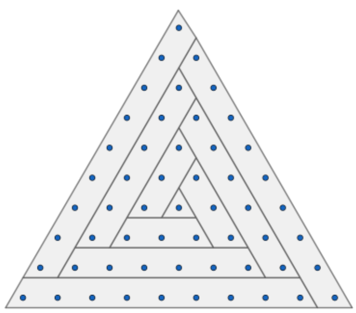


T_{3n+2}

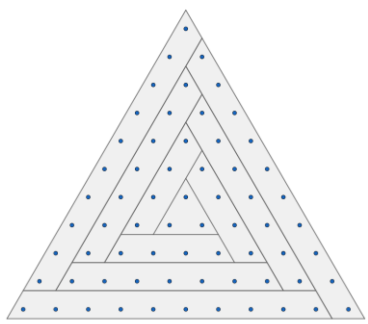
$3(T_n + S_{n+1})$ [3.28]
$3(2T_n + T_{n+1})$ [3.29]



T_{3n}



T_{3n+1}



T_{3n+2}

$\boldsymbol{T_{3(n-1)} + 3Tr_{3n-1,3n-2}}$ **[3.18]** $\boldsymbol{T_{3n-2} + 3Tr_{3n,3n-1}}$ **[3.24]** $\boldsymbol{T_{3n-1} + 3Tr_{3n+1,3n}}$ **[3.30]**

$\boldsymbol{T_{3(n-1)} + 3(T_{3n-1} - T_{3n-2})}$ **[3.19]** $\boldsymbol{T_{3n-2} + 3(T_{3n} - T_{3n-1})}$ $[3.25]$ $\boldsymbol{T_{3n-1} + 3(T_{3n+1} - T_{3n})}$ **[3.31]**

(f) Square triangle numbers

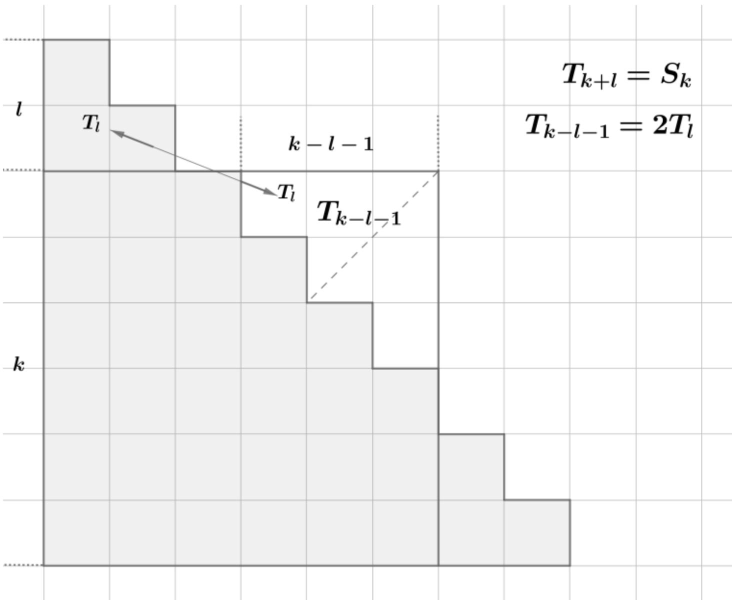
Of interest is the question 'When is one triangle number a multiple of another?' In particular, 'When is one triangle number *twice* another?' (The following derivation is due to Mark Bennet, Mathematics Stack Exchange 29.12.16.)
$T_m = 2T_n$
$\Leftrightarrow m(m+1) = 2n(n+1)$
$\Leftrightarrow 4m^2 + 4m = 8n^2 + 8n$
$\Leftrightarrow (2m+1)^2 = 2(2n+1)^2 - 1$ (completing the square),
a statement of the form
$M^2 = 2N^2 - 1$, a Pell equation, whose solution sets are the pairs of 'side diameter' numbers.
The first (M, N) pairs are (7, 5), (41, 29), corresponding to the respective (m, n) pairs (3, 2), (20, 14).

The bonus is that, potentially, we have answers to the question 'When is a triangle number a square?' because, as the next figure shows, the condition $T_{k+l} = S_k$ is equivalent to the condition $T_{k-l-1} = 2T_l$.

We have:

$l = n$,
$k = l + m + 1 = n + (m+1)$,
$k + l = 2n + (m+1)$,
$T_{2n+(m+1)} = S_{n+(m+1)}$.

The (m, n) pairs above give us respectively $T_8 = S_6, T_{49} = S_{35}$.



(g) What about *cube* triangle numbers?

The number of triangle numbers which are *squares* is infinite but there are no cubes > 1. The following argument comes from Wolfram MathWorld.

We have:

$T_n = C_m$,
$\frac{n(n+1)}{2} = m^3$,
$4n(n+1) = 8m^3 = (2m)^3$,
$(2n+1)^2 - (2m)^3 = 1$.
By a confirmed conjecture of Catalan, which is therefore a theorem, the only pair of perfect powers differing by 1 are 3^2 and 2^3 so, uniquely, $n = m = 1$.

An aside: The conjecture was proved for all odd values of $(2n+1)$. If, in place of $(2n+1)$, we specify a prime, a, and in place of $(2m)$ we specify a prime c, and, in place of 1, we specify some integer b, the Catalan theorem is established as follows.

$a^2 - c^3 = b^2$,
$a^2 - b^2 = c^3$,
$(a+b)(a-b) = c^3$,
The right side can only be rendered as the two unequal factors: c, which must equal the smaller bracket on the left, and c^2, which must equal the larger bracket on the left:
$a - b = c$,
$a + b = c^2$.
Adding,
$2a = c(c+1)$.
The solution $c = a$ would imply $c = 1$, which is not a prime.
Hence the solution $c = 2, a = 3, b = 1$, which is unique.

Note also $a = \frac{c(c+1)}{2} = T_c$, the only prime triangle number.

(B) Numerical properties

(a)

The odd number p divides every p^{th} integer on the number line. Inspecting the factors in the numerator of T_n:

(1)(2)
(2)(3)
(3)(4)
...
$(np-1)(np)$
$(np)(np+1)$
...

we see:
(i) that here too the odd number p divides every p^{th} integer on the number line,
(ii) that p divides both T_{np-1} and T_{np},
(iii) that p divides the second bracket in the numerator of T_{np-1} and the first in the numerator of T_{np}.

We can use these three lemmas to solve problems of the following kind.

We are given six consecutive triangle numbers:

$T_a \quad T_{a+1} \quad T_{a+2} \quad T_{a+3} \quad T_{a+4} \quad T_{a+5}$

and told that 5 divides T_a and T_{a+4}. What can we infer?

If we were only told that 5 divides T_{a+4}, we would be in doubt as to whether $a+4$ was of the form $5n-1$, in which case 5 would divide T_{a+5}, or $5n$, in which case 5 would divide T_{a+3}. But, since 5 divides T_a, and every 5^{th} triangle number divides by 5, we know that 5 divides T_{a+5}.

A more elaborate example:

We are given four consecutive triangle numbers under 150 and tasked with identifying them, given the following information:

(1) 3 does not divide T_a,
(2) 3 divides T_{a+1},
(3) 5 divides T_{a+2},
(4) 11 divides T_{a+3}.

Inferences:

From **(1), (2)**: Since 3 divides T_{a+1} but not T_a, it must divide T_{a+2}. **(5)**
From **(3), (5)**: $3 \times 5 = 15$ divides T_{a+2}. **(6)**
From **(4), (6)**: If 11 divided T_{a+2} as well as T_{a+3}, T_{a+2} would be $\geq 11 \times 15$,
which is > 150, so it does not. **(7)**
From **(4), (7)**: since 11 divides T_{a+3} but not T_{a+2}, it must divide T_{a+4}. Furthermore, it must be a divisor of the smaller bracket in the expression for T_{a+4}, $\frac{(a+4)(a+5)}{2}$.
We have: $a+4=11k$. The smallest solution, $a=7$, gives a value for T_{a+4} of 66, < 150.
The next, $a=18$, gives a value > 150.
Therefore $T_{a+4}=66=\frac{11\times12}{2}$, and we can derive from it the values required:
$T_{a+3}=\frac{10\times11}{2}=55$,
$T_{a+2}=\frac{9\times10}{2}=45$,
$T_{a+1}=\frac{8\times9}{2}=36$,
$T_a=\frac{7\times8}{2}=28$.

(b)

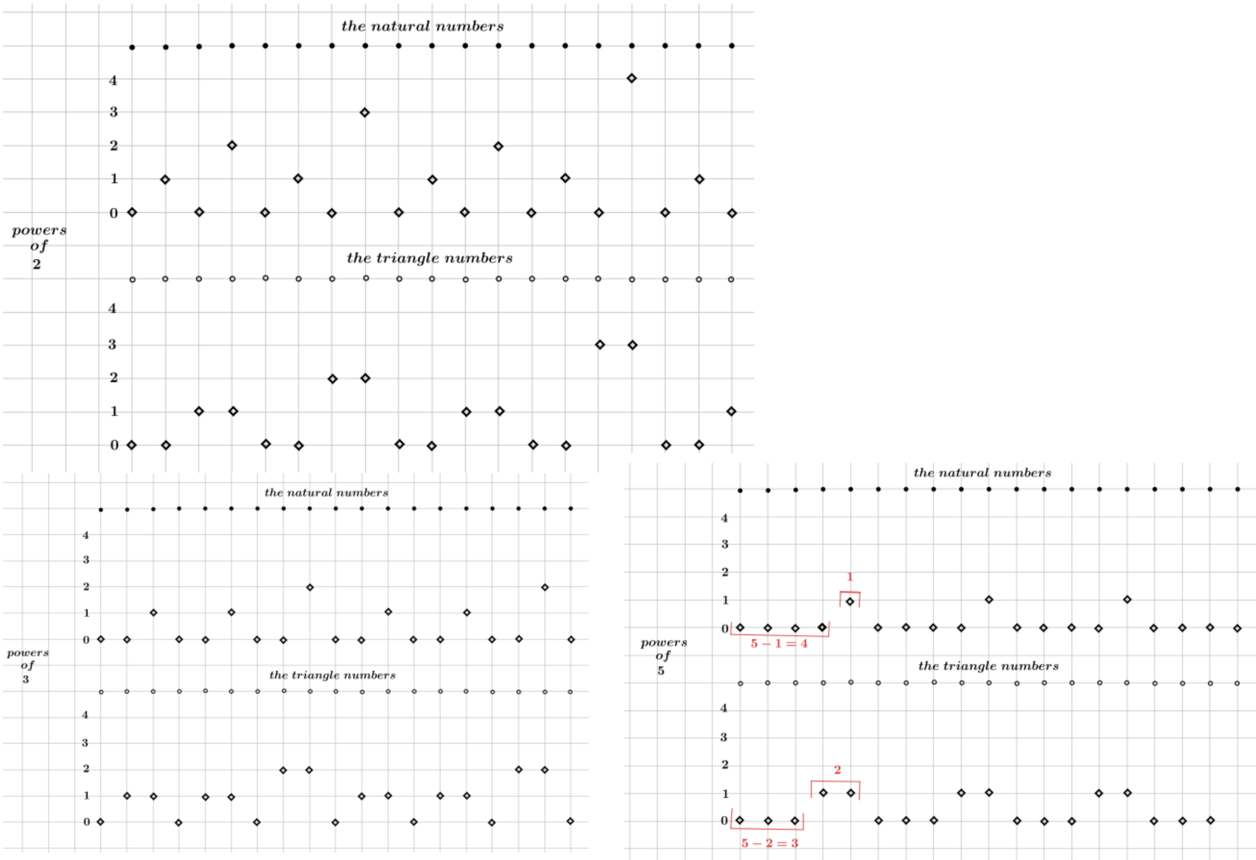
Continuing our analysis of the patterns revealed in **(a)**, it's interesting to see how, when considering particular primes, the prime factorisation patterns of the sequence of triangle numbers compare with those of the natural numbers. We can think of the comparison in two ways:

(i) the natural numbers sampled at positions 1, 3, 6, 10, ... :

(ii) the consequences of the formula $\frac{n(n+1)}{2}$ for the individual triangle numbers.

The figures following show how the powers respectively of 2, 3 and 5 appear. Notice:

(i) Compared with the factors of the natural numbers, every entry for an exponent > 0 is duplicated in the triangle numbers.
(ii) For an odd prime p, the number of consecutive entries for the exponent 0 is $(p-1)$ for the natural numbers, $(p-2)$ for the triangle numbers.

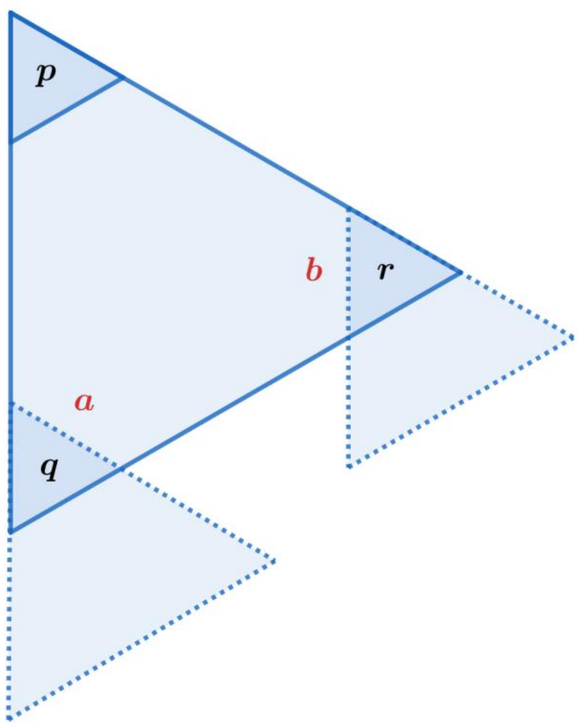


(c)

It's instructive to draw diagrams showing sequences of partitions of the triangle numbers into two parts: $T_p = aT_q + bT_r\ , q > r, a > b \geq 0$. This equation serves for a number of the identities we've met. But we have chosen in particular:

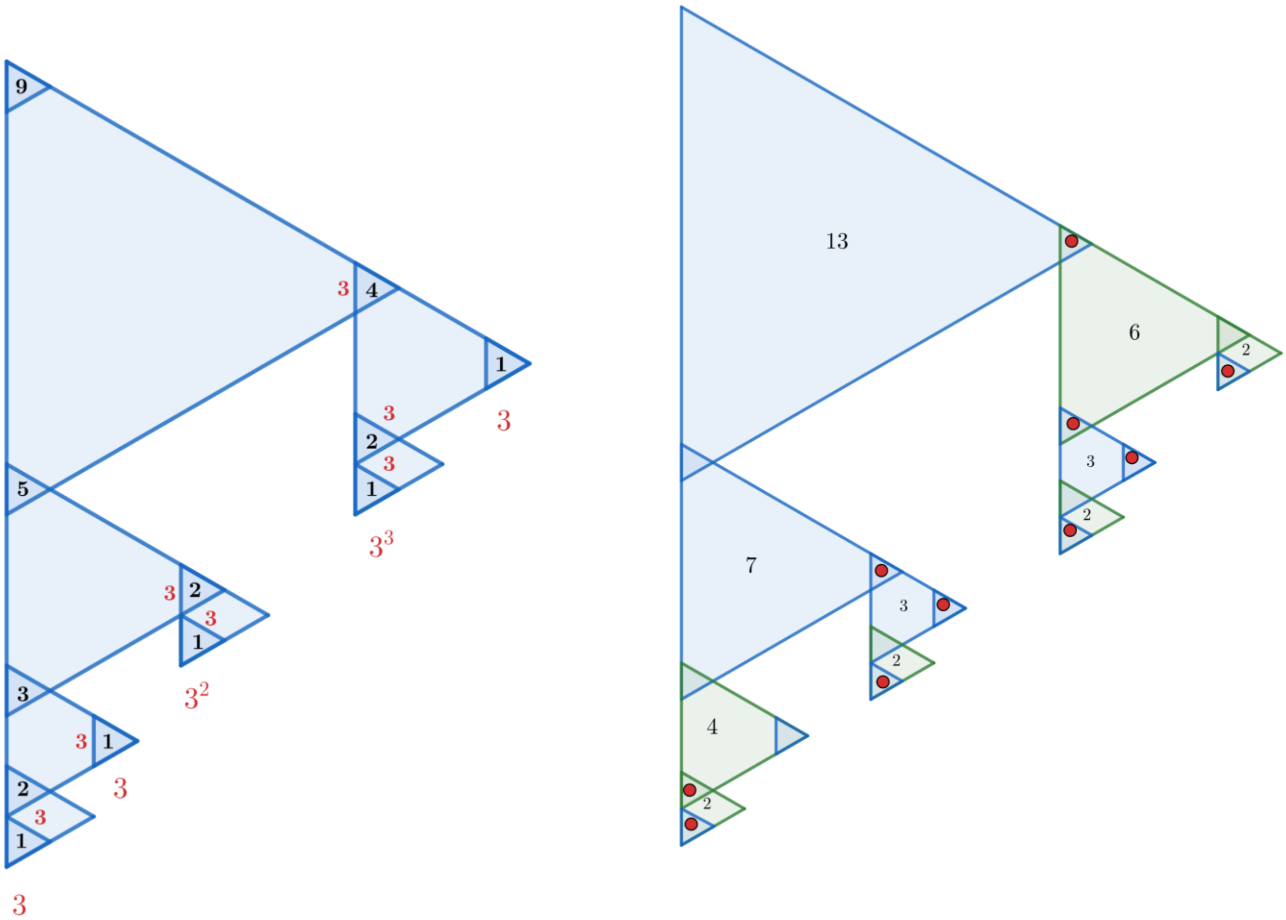
$T_{2n} = 3T_n + T_{n-1}$ **[3.9]** and
$T_{2n+1} = 3T_n + T_{n+1}$ **[3.8]**.

In an equilateral triangle of side length p, we enter p at the apex, q at the left base vertex, r at the right base vertex, and the coefficients a and b inside those.
q and r in turn become the apices of new triangles of the same kind. The process terminates when the last q value is 1.



This dichotomy will always be possible since every triangle number suffix is either odd or even.

In the figure for T_9 below left the unit coefficients are not marked. We have computed the total number of T_1s by multiplying the relevant coefficients, whose total must equal $T_9 = 45$. The figure below right makes the point that the position of the coefficient '3' occurs on the left when the T suffix is even, on the right when it is odd.

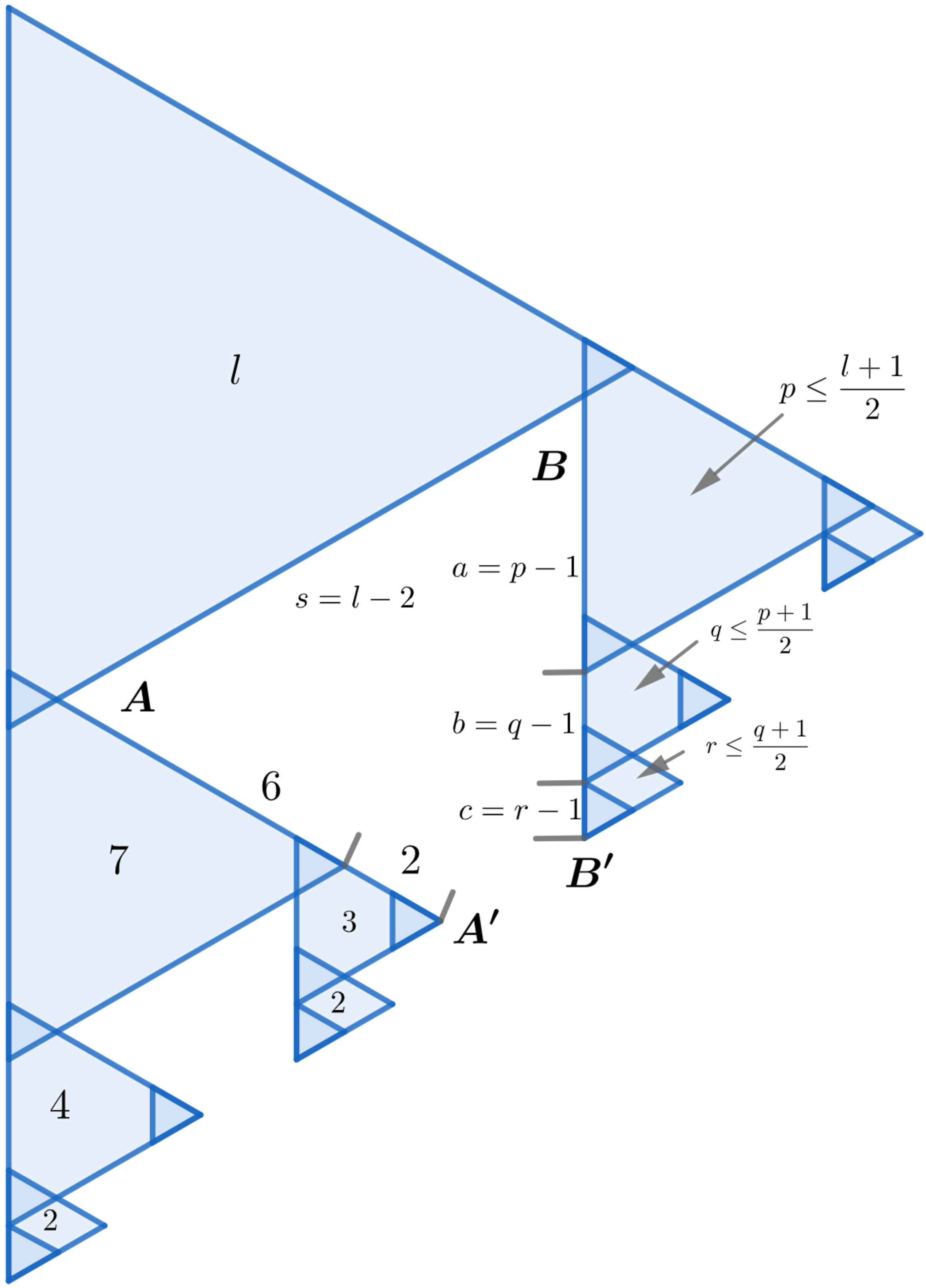


We need to show (see next figure) that the segment *AA'* does not cross *BB'*. This is equivalent to showing that $a + b + c + \cdots \leq s$.

Since $p \leq \frac{l+1}{2}$, $a = p - 1 \leq \frac{l-1}{2} = \frac{s+1}{2}$.
Since $q \leq \frac{p+1}{2}$, $b = q - 1 \leq \frac{p-1}{2} = \frac{s+1}{4}$.
Since $r \leq \frac{q+1}{2}$, $c = r - 1 \leq \frac{q-1}{2} = \frac{s+1}{8}$.
… … … .

Thus $a + b + c + \cdots \leq \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)(s + 1) = s + 1$.

Now, our convention of putting the greater of q, r, namely q, on the left, ensures that, along a given edge like *BB'*, we cannot encounter two consecutive cases where the '$\leq$' sign resolves to an '$=$' sign. Thus, for ≥ 2 cases (the minimum), $a + b + c + \cdots < s + 1$, i.e. $a + b + c + \cdots \leq s$, as required.



l
$p \leq \frac{l+1}{2}$
$\boldsymbol{B}$
$a = p - 1$
$s = l - 2$
$q \leq \frac{p+1}{2}$
$\boldsymbol{A}$
$b = q - 1$
$r \leq \frac{q+1}{2}$
6
$c = r - 1$
7
2
$\boldsymbol{B'}$
3
$\boldsymbol{A'}$
2
4
2

Because the tree is a dichotomy, descending from any given apex p, the number of daughter triangles goes up as a power of 2. For example, starting with $p = 100$, we have the breakdown and totals shown in the table below.

Observing the structure, which distributes the coefficient 3 to one side, 1 to the other, the ***true*** number of daughter triangles goes up as a power of 2^2.

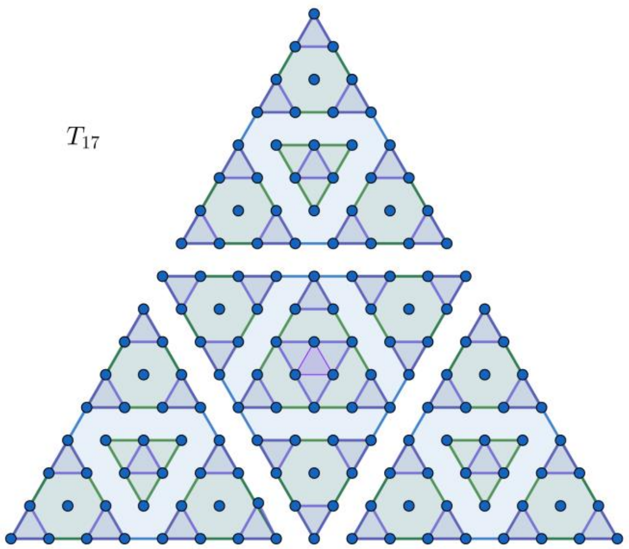
If we define daughter triangles to the triangle with apex number p as belonging to the 1^{st} generation, granddaughter triangles as belonging to the 2^{nd}, and so on, the total value of all triangle numbers represented by triangle apices of the same generation, is that of the apex triangle number, T_p. So we have the equalities of the kind shown in the third column of the table below.

Number of triangles in the triangle -gram	Breakdown
2^0	1(100)
2^1	1(50) 1(49)
2^2	2(25) 2(24)
2^3	2(13) 4(12) 2(11)
2^4	2(7) 8(6) 6(5)
.	.
.	.
.	.

True number of triangles	Breakdown of apex values	Breakdown of triangle number totals
2^0	1(100)	T_{100} =
2^2	3(50) 1(49)	$3T_{50} + T_{49}$ =
2^4	10(25) 6(24)	etc.
2^6	10(13) 48(12) 6(11)	etc.
2^8	10(7) 180(6) 66(5)	etc.
.	.	.
.	.	.
.	.	.

Since a triangle number is either of the form $3n$ or $3n + 1$, descending from a given apex p all the way down to the terminating vertices, those with apex value 1, the total number of those vertices must be a number of one of the two kinds since $T_n = T_n T_1$.

A more compact representation of **[3.8]** and **[3.9]** partitions an equilateral triangle dot figure iteratively. Here for example is T_{17}:



(d) Though we shall not prove this, with triangle numbers, every equality is a special case of an identity or a set of nested identities. Taking **[3.12]** as our example, which is just a special case of **[3.1]**, we have, schematically:

$T_{S_n} = \quad S_n \quad + T_{S_n - 1}$ $\qquad T_9 = S_3 + T_8$

[3.5]

$T_{n-1} \qquad T_n$

$\boldsymbol{n = 2k + 1}$

$T_{2k} \qquad T_{2k+1}$ $\qquad T_9 = T_2 + T_3 + T_8$

[3.9] | **[3.8]**

$T_{k-1} \mid 3T_k \qquad 3T_k \mid T_{k+1}$

$\boldsymbol{k = 1}$

$(T_0) \mid 3T_1 \qquad 3T_1 \qquad T_2$

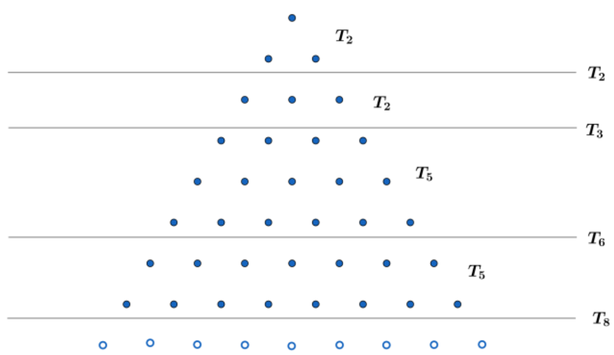
[3.9] $\qquad (T_0) \mid 3T_1$

T_2 $\qquad$ **[3.9]**

T_2

$3T_2$ $\qquad T_9 = 3T_2 + T_8$

In a special case of identity **(ii)** in part **(b)**, $a = b = 1$, so that $T_p = T_q + T_r$. Note that the right side is symmetrical in q and r. We can form chains in this manner:



The triangle numbers between the lines appear as trapezoids.

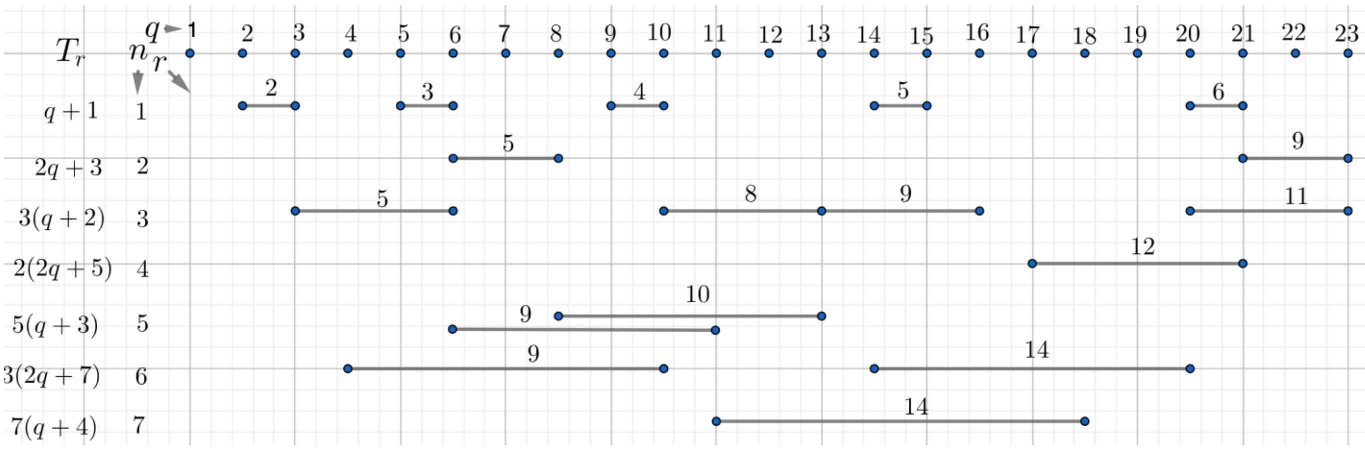
The algebraic symmetry allows us to write the same equality in two ways, for example:

$T_5 + Tr_{8,5} = T_8$,
$T_6 + Tr_{8,6} = T_8$.

If, in our equation $T_p = T_q + T_r$, and we write p as $q + n$, T_r represents the difference between two triangle numbers n positions apart in the sequence of triangle numbers. We have:

$$T_r = \frac{r(r+1)}{2} = \frac{(q+n)(q+n+1) - q(q+1)}{2} = \frac{n(n+2q+1)}{2}.$$

Here are the first few examples we encounter as n, q range over the natural numbers while satisfying the above identity:



(e)

If $T_p = T_q + T_r$, or in graphic form, $P = Q + R$, we can ask if there's a general method for dissecting one side of the equation into the other. This will not necessarily be the most economic dissection in terms of the number of pieces but it will be universally applicable. The following iterative procedure fits the bill. We use the 'staircase' forms.

1. Lay *R* over *Q* so that the resulting stepped hypotenuse corresponds to that of *P* **.**

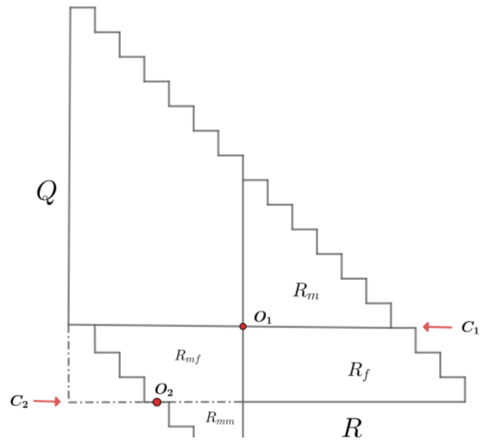
2a. Make cut *C1*, producing two pieces from *R*: R_f (*f* for 'fixed') and R_m *m* for 'moveable').

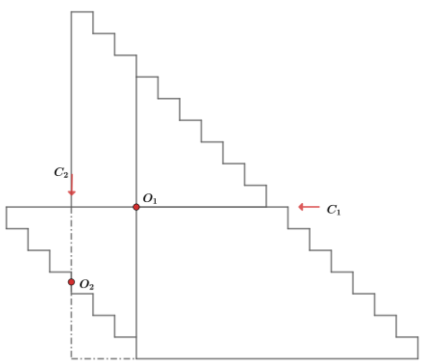
2b. Give R_m a half-turn about O_1.

3a. Make cut *C2*, producing two pieces From R_m: R_{mf}, R_{mm}.

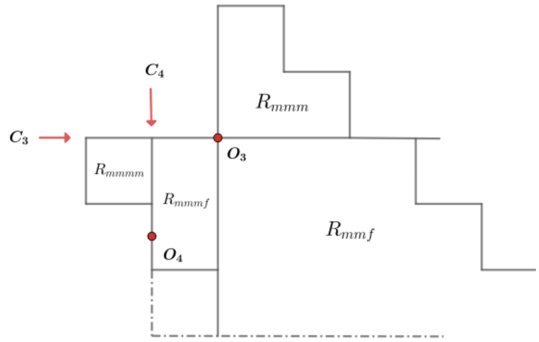
3.b. Give R_{mm} a half-turn about O_2.

The algebraic symmetry tells us that we can swap *Q* and *R*:





In the example illustrated, this marks the end of the process: the *P* outline is filled by pieces constituting *Q* and *R*. But in the general case, we must repeat steps **a** and **b** until this is achieved. Here for example are the final stages in the case $p = 23, q = 20, r = 11$:

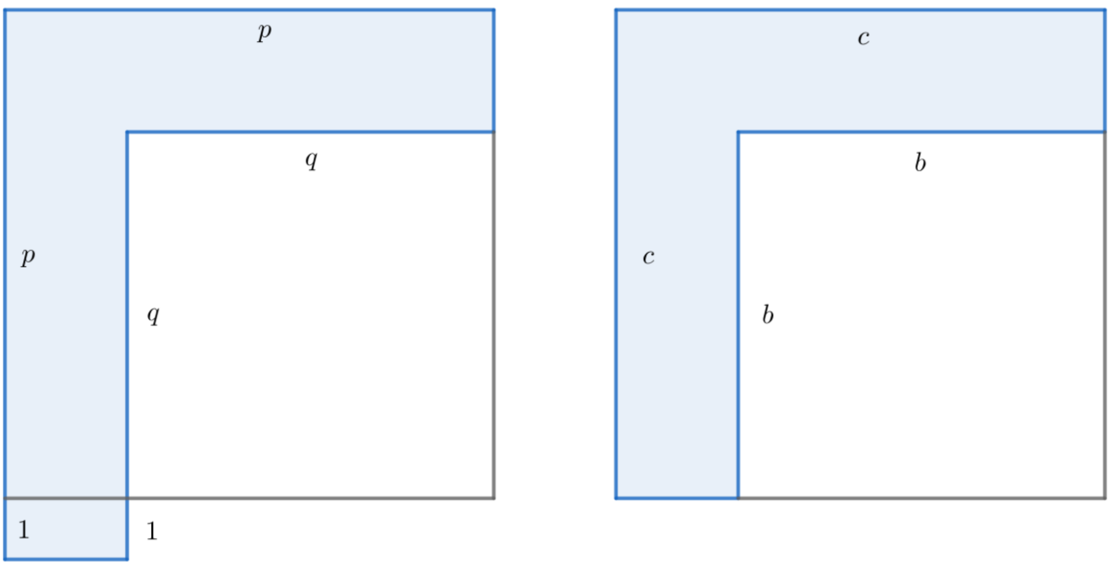


The validity of the method rests on the fact that, by means of half-turns about vertices or edge midpoints of a square lattice, a given square A can be moved to coincide with a chosen square B anywhere in the plane.

(f)

An investigation worth pursuing is the analogy between the Pythagorean triples, (a, b, c), where the general term x represents x^2 in the identity, and the triangle triples (p, q, r), where x represents $\frac{x(x+1)}{2}$.

This figure characterises the distinction:



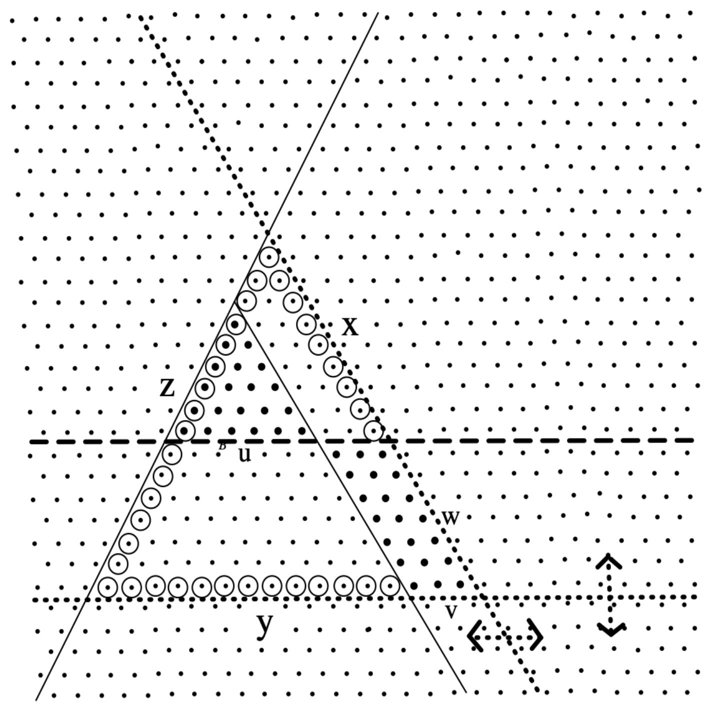
$(p + q + 1)(p - q) = r(r + 1)$ $\qquad$ $(c + b)(c - b) = a^2$

In the same way we can class cases among the Pythagorean triples, e.g. those where the hypotenuse and a leg differ by 1, we can class cases among the triangles, e.g. those distinguished by an n value of 1 on the chart in part **(e)**.

a	b	c:	
S_r	S_{2b}	S_{2b+1}	For odd $r > 1$, put $b = \frac{r^2-1}{4}$. Then $S_{\frac{r^2+1}{2}} = S_{\frac{r^2-1}{2}} + S_r$.
r	q	p	For $r > 1$, put $q = T_r - 1$. Then $T_{T_r} = T_{T_r-1} + T_r$.
T_r	T_q	T_{q+1}	

Whereas there is one set of parametric equations which produces all Pythagorean triples, for triangles we need a separate set for each type, requiring four parameters in all. Failing that, here is a nomogram we can use.

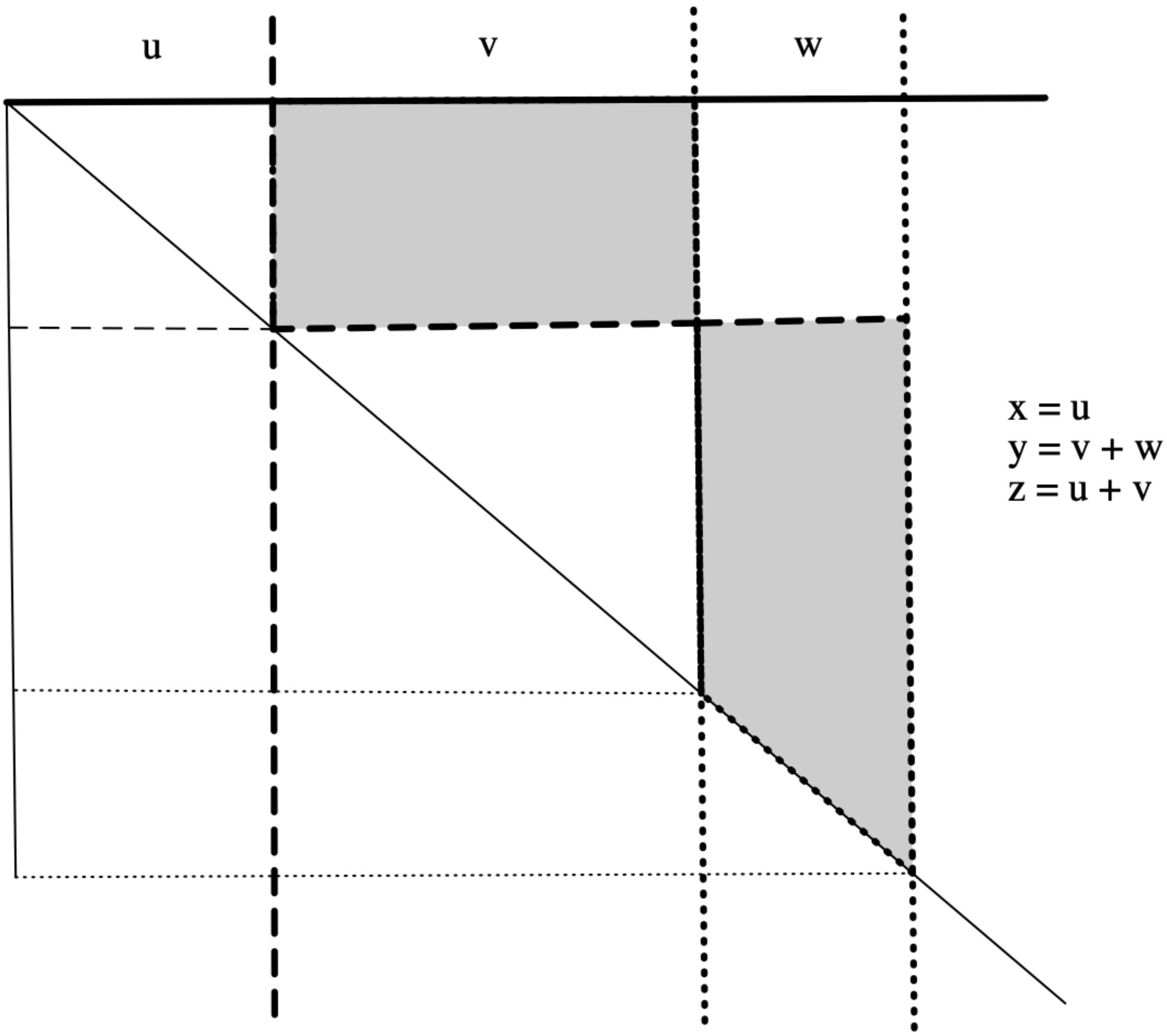
With the two solid lines fixed, we choose a position for the dashed line and adjust the two dotted lines until the number of dots in the parallelogram equals the number in the small triangle (shown by the big dots in the figure beneath). We then read off the number of dots on the sides of the smaller triangles, *x* and *y*, and of the large triangle, *z*, to give a solution of $T_x + T_y = T_z$.



Here's why it works.

Readers will see that the task in figurate terms is to find a gnomon to a triangle (the large trapezium) which is itself triangular. We must be able to dissect this trapezium into a smaller trapezium and a parallelogram, and dissect the parallelogram into a triangle to which the smaller trapezium is a gnomon. This is what the nomogram achieves.

We have $x = u + v$, $y = u + w$, $z = u + v + w$. By taking every u value and every factorisation $T_u = vw$, we generate all possible solutions. The figure below shows an analogous construction for the equation $S_x + S_y = S_z$, where S_n is the *n*th square number, n^2. Solutions are restricted by the Pythagorean condition, namely, the gnomon to a square S_x is itself a square S_y if and only if *x*, *y* form the shorter sides of a right triangle with integral hypotenuse *z*. In the triangular case by contrast we have a range of obtuse-angled triangles.



(g) Now consider for comparison sequences of three consecutive numbers of each type, squares and triangles.

We can show that three consecutive squares never sum to a square. Let $S_{n-1} + S_n + S_{n+1} = S_a$. Forming a quadratic equation in *a* leads to:

$a^2 = 3n^2 + 2 = 2$ (mod 3).
But a^2 = 0 or 1 (mod 3).
Therefore the equation has no solution in integers.

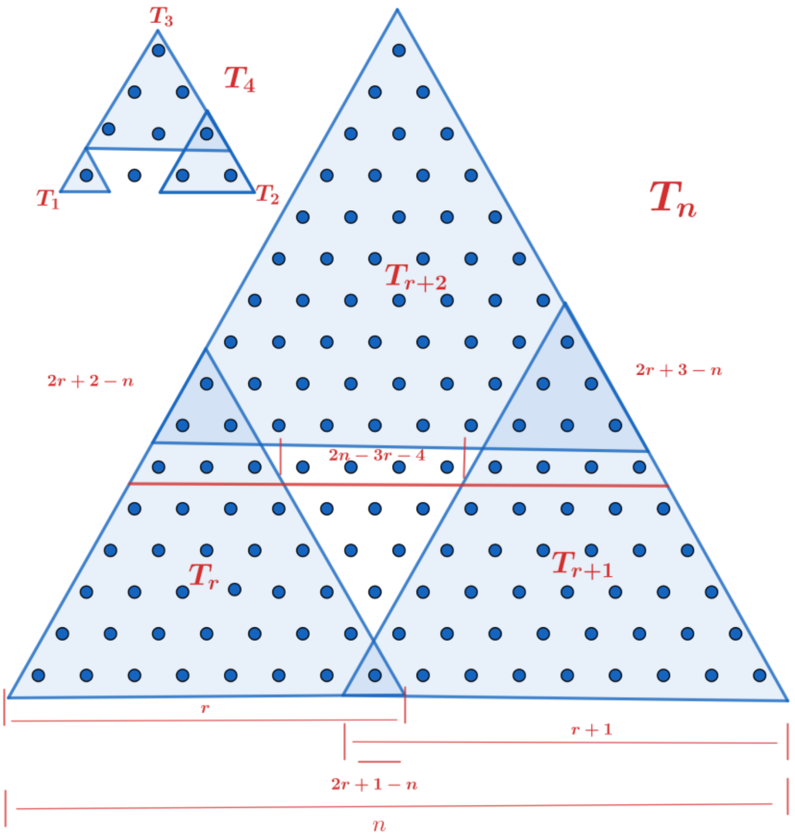
Certainly 3 consecutive *triangles* can sum to a *square*: $T_5 + T_6 + T_7 = S_8$. Indeed, it turns out there is an infinite number of such sets. The squares, a_n, are generated by this recurrence relation:
$a_n = 10a_{n-2} - a_{n-4}$ with starting values $a_1 = 1, a_2 = 2, a_3 = 8, a_4 = 19$. We can use **[3.7]** in the form $T_r + T_{r+1} + T_{r+2} = 3T_{r+1} + 1$ to find the triangles. For example:
$19^2 = 361 = 3(120) + 1.$
$120 = \frac{s(s+1)}{2},$
$s^s + s - 240 = 0,$
$s = \frac{-1+\sqrt{1+960}}{2} = 15,$
giving $T_{14} + T_{15} + T_{16} = S_{19}$.

We can find an infinite number of sets of three consecutive *triangles* which sum to a *triangle*. Here are the smallest two:

$T_1 + T_2 + T_3 = 3T_2 + 1 = T_4$,
$T_8 + T_9 + T_{10} = 3T_9 + 1 = T_{16}$.

How do we find such sets, $T_r + T_{r+1} + T_{r+2} = T_n$? **[1]**

In the following figure we have drawn a triangle for T_n containing T_r, T_{r+1}, T_{r+2}. For equality, it must be the case that the regions of overlap sum to the central uncovered region. We can find the dimensions of these from the figure, leading to the reduced equation $T_{2r+1-n} + T_{2r+2-n} + T_{2r+3-n} = T_{2n-3r-4}$. **[2]**
We have in effect converted a two-dimensional problem to a 1-dimensional one.



Entering new values in the suffix of a term in **[2]**, we obtain the old value as the suffix of the corresponding term in **[1]**:

we have:

(1)$r_p = 2r_{p+1} + 1 - n_{p+1}$,
(2) $n_p = 2n_{p+1} - 3r_{p+1} - 4$.

Combining the two equations, we have:

$r_{p+1} = n_p + 2r_p + 2$,
$n_{p+1} = 2n_p + 3r_p + 5$.

With starting values
$r_0 = 1, n_0 = 4$,
this is the relation we require.

Here is a table of the first few (r, n) pairs generated this way.

r_p	n_p	r_{p+1}	n_{p+1}
1	4	8	16
8	16	34	61
34	61	131	229
131	229	493	856

Using **[3.23]**, we obtain the following equation, which the values produced by the recurrence relation must satisfy:
$T_{3k+4} = 3T_l - 2$.
So we have for example these specific equalities:
$T_7 = 3T_4 - 2$,
$T_{28} = 3T_{16} - 2$.

(h) The triangle numbers modulo 3

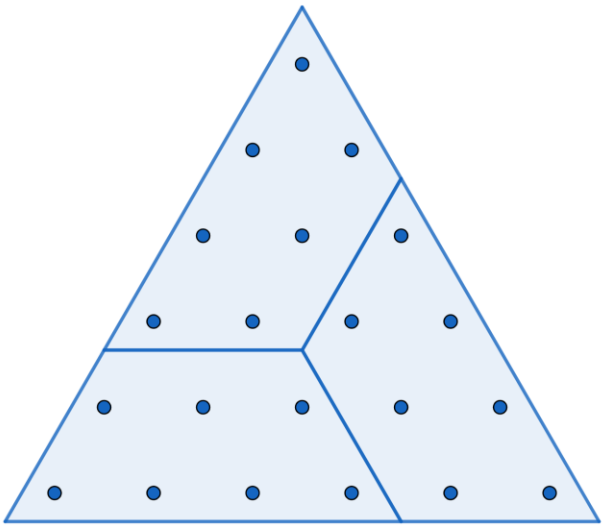
We have:

$T_{3n} = 0$ (mod 3),
$T_{3n+1} = T_{3n} + (3n + 1) = 1$ (mod 3),

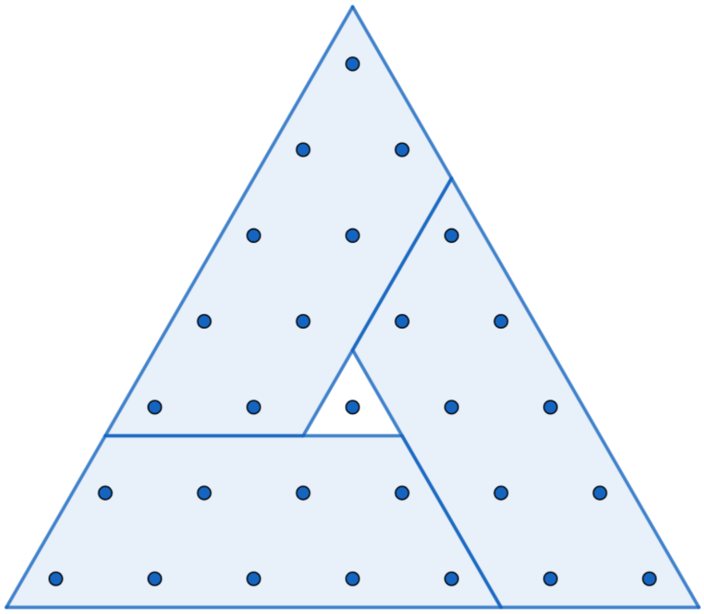
$T_{3n+2} = T_{3n+1} + (3n + 2) = 0 \text{ (mod 3)}.$

So, modulo 3, the numbers just cycle like this: 1 0 0 1 0 0 1 0 0 ...

We see this also from their figurate representation. Either there is a dot in the centre or there is not:



$\boldsymbol{T_6 = 3Tr_{4,2}}$



$\boldsymbol{T_7 = 3Tr_{5,3} + 1}$

(C) Algebraic properties

The form of the defining equation, $T_n = \frac{n(n+1)}{2}$, gives rise to interesting properties.

(i) We have: $n(n + 1) = 2T_n$, $n^2 + n - 2T_n = 0, n = \frac{-1+\sqrt{8T_n+1}}{2}$. T_n will be a triangle number iff the expression under the square root sign is a square. (We recognise it from **[3.6]**.)

(ii) Since n and $(n + 1)$ are consecutive integers, they share no prime factors. If we are given a reasonably small T_n, say $120 = 2^3 \times 3 \times 5$ so that $2T_n = 2^4 \times 3 \times 5$, we can easily sort the products so that one exceeds the other by 1: $3 \times 5 = 2^4 - 1$; $n = 15$. If, on the other hand, T_n is large, we must use **(i)**.

(iii) Chaining the triangle numbers in a product so that the numerator brackets run consecutively, we have

$(2n)! = 2^n \prod_{i=1}^{i=n} T_{2i-1}$ **(A)** and

$(2n + 1)! = 2^n \prod_{i=1}^{i=n} T_{2i}$ **(B)**.

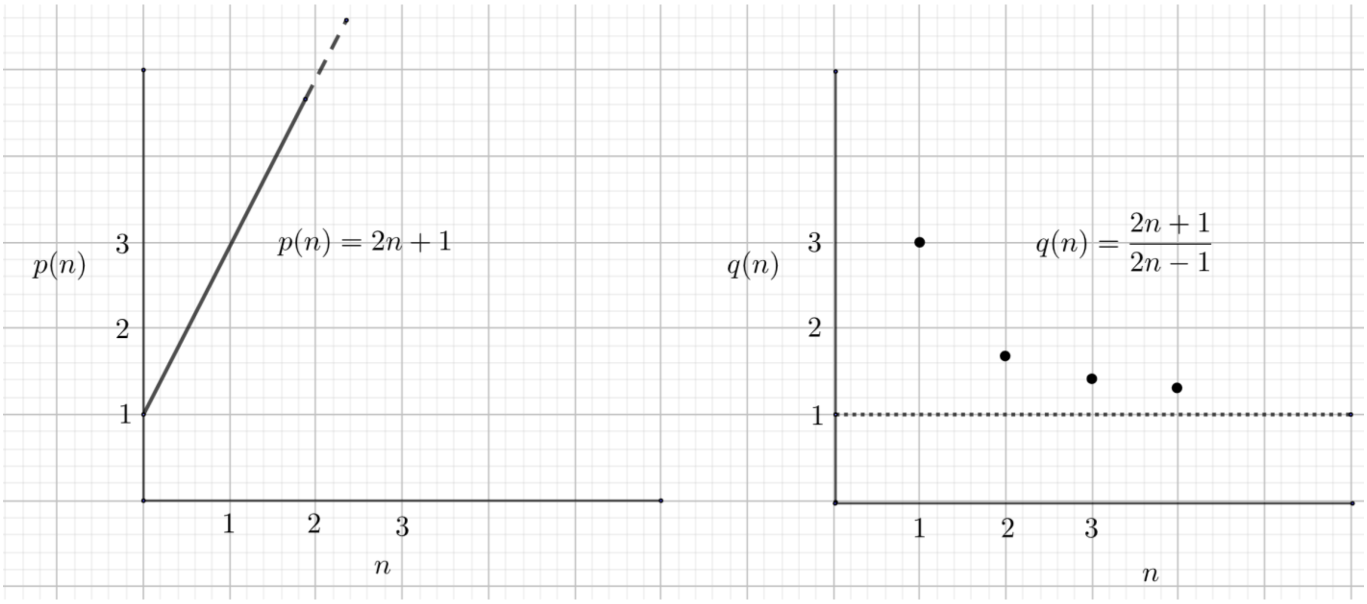
Dividing **(B)** by **(A)**, we have $\frac{(2n+1)!}{(2n)!} = 2n + 1 = \prod_{i=1}^{i=n} \frac{T_{2i}}{T_{2i-1}}$.

To see why the product telescopes, we only have to spell out the product and observe the diagonal pattern of cancellation:

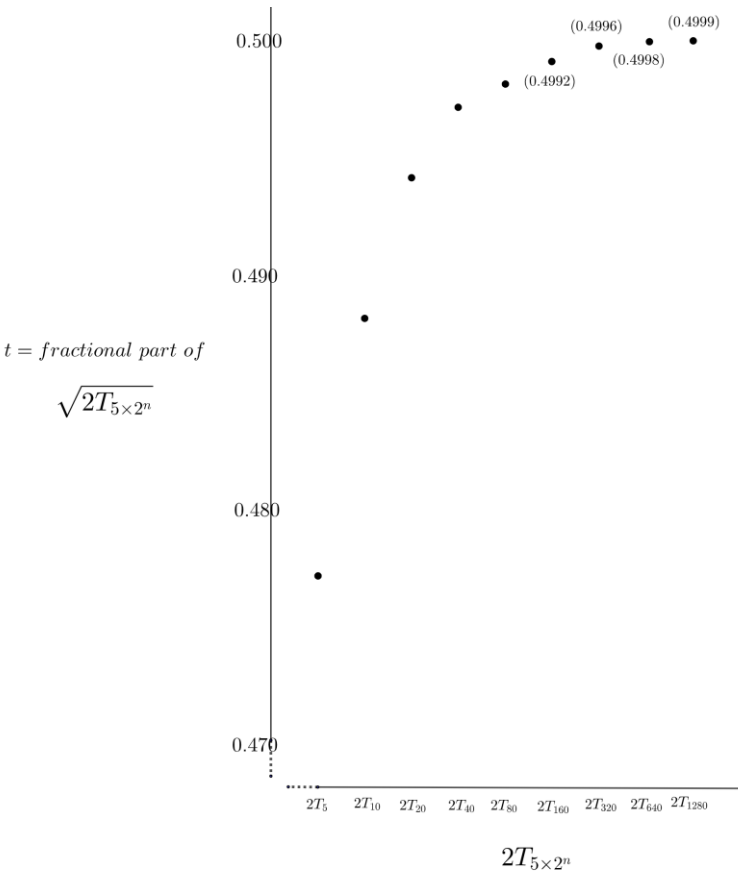
$$\frac{T_2}{T_1} \times \frac{T_4}{T_3} \times \frac{T_6}{T_5} \times \ldots = \frac{\cancel{2} \times 3}{1 \times \cancel{2}} \times \frac{4 \times 5}{3 \times 4} \times \frac{6 \times 7}{5 \times 6} \times \ldots \times \frac{\cancel{2n} \times (2n+1)}{(2n-1) \times \cancel{2n}} = 2n + 1.$$

The product p then is a linear function of the number of terms, n: $p(n) = 2n + 1$. Not so the n^{th} term,

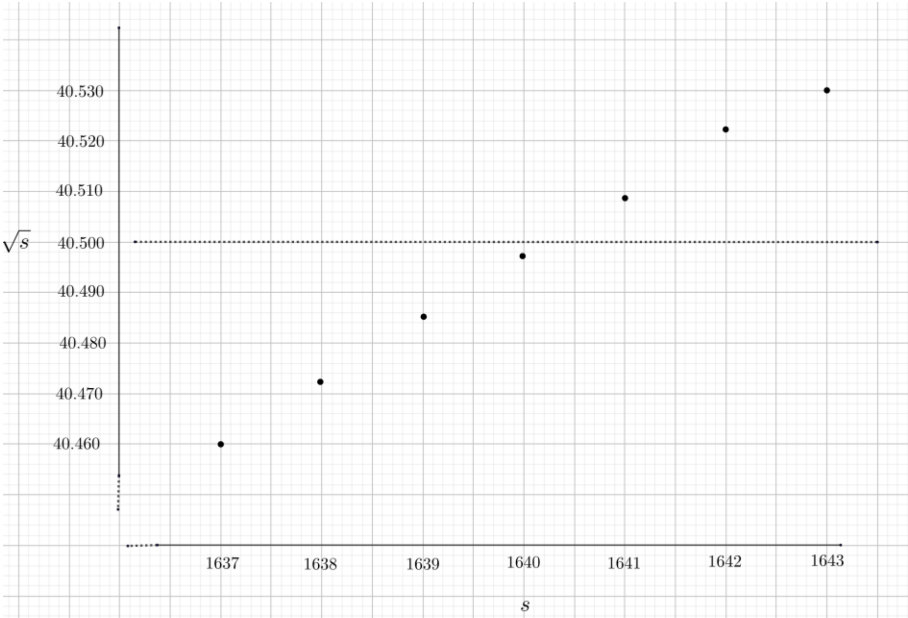
$q(n) = \frac{2n+1}{2n-1}$. Compare their respective graphs. $q(n) \to 1$ as $n \to \infty$.



The pronic number $s = n(n+1)$ is already near to a square. What happens if we take the square root? We have $\sqrt{s} = \sqrt{(n+\frac{1}{2})^2 - \frac{1}{4}} = n + t$. As $n \to \infty$, $t \to \frac{1}{2}$. The convergence is rapid:



Given a set of integers spanning a range between squares which contains a pronic number *p*, we take square roots (here shown to 3 decimal places). $(p+1)$ is the first number for which $t > \frac{1}{2}$, identifying *p*.



We can show this as follows.

$s_0 = k^2, s_1, s_2, \ldots. s_{2k-1}, s_{2k}, s_{2k+1} = (k+1)^2$ is a sequence of consecutive positive integers. For a particular consecutive pair, s_n, s_{n+1},

$$\sqrt{s_n} = k + t_n,$$
$$\sqrt{s_{n+1}} = k + t_{n+1},$$

where *k* is the integer part, t_n, t_{n+1} the respective fractional parts of $\sqrt{s_n}$, $\sqrt{s_{n+1.}}$
If $t_n < \frac{1}{2}$, $t_{n+1} > \frac{1}{2}$, show that $s_n = n(n+1)$.

Between s_0 and s_{2k+1} there will be a single number $s_u = k(k+1)$.

$$\sqrt{s_u} = \sqrt{\left(k+\frac{1}{2}\right)^2 - \frac{1}{4}} = \left(k+\frac{1}{2}\right) - t_x = k + \left(\frac{1}{2} - t_x\right),$$
$$\sqrt{s_{u+1}} = \sqrt{\left(k+\frac{1}{2}\right)^2 + \frac{3}{4}} = \left(k+\frac{1}{2}\right) + t_y = k + \left(\frac{1}{2} + t_y\right).$$

$$\frac{1}{2} - t_x = t_u < \frac{1}{2},$$
$$\frac{1}{2} + t_y = t_{u+1} > \frac{1}{2}.$$

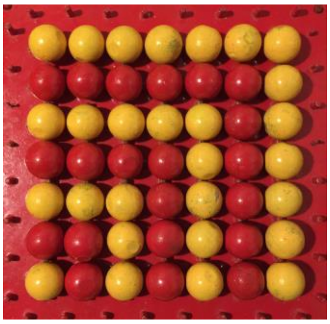
For all the integers < s_u the *t* value will be < $\frac{1}{2}$.
For all the integers $>$ s_u the *t* value will be $> \frac{1}{2}$.

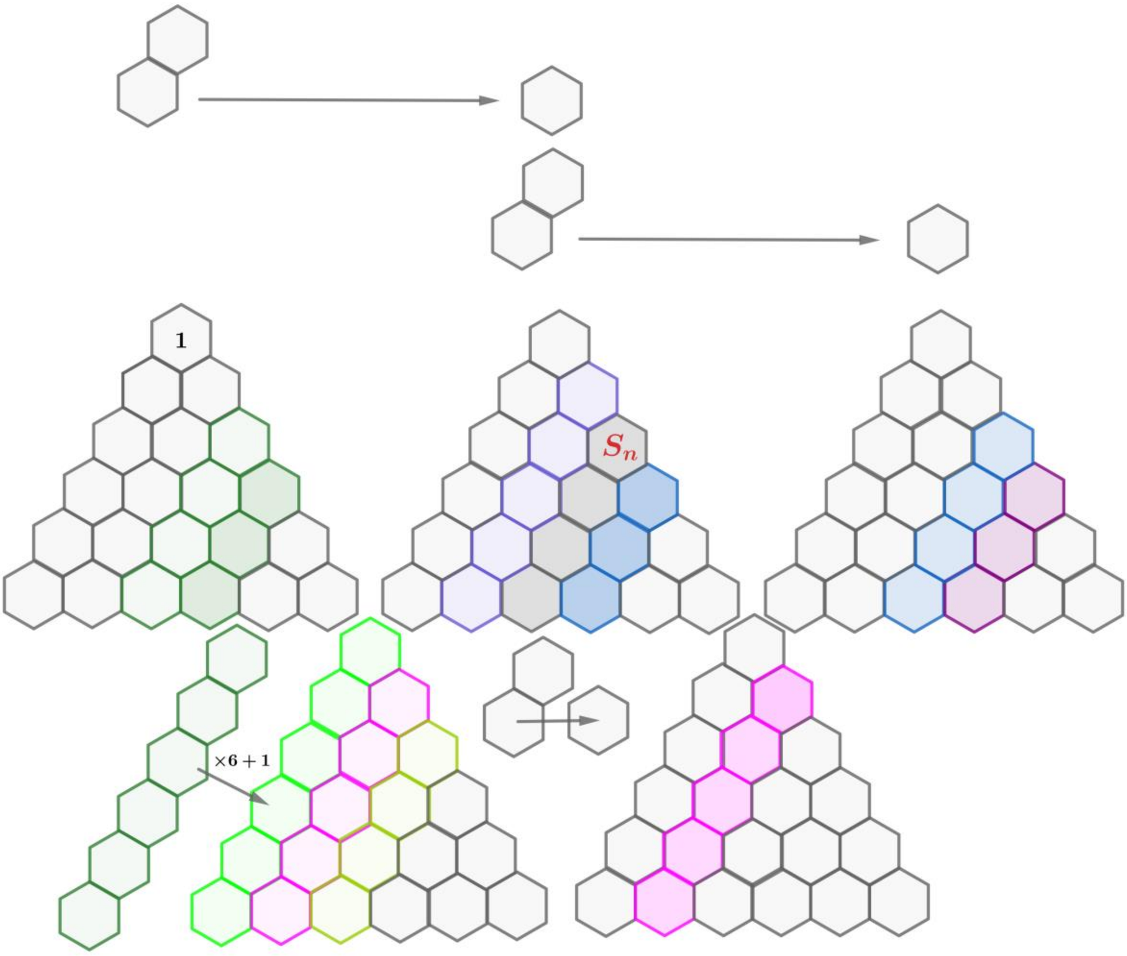
This identifies *u* uniquely as *n*.

The result follows.

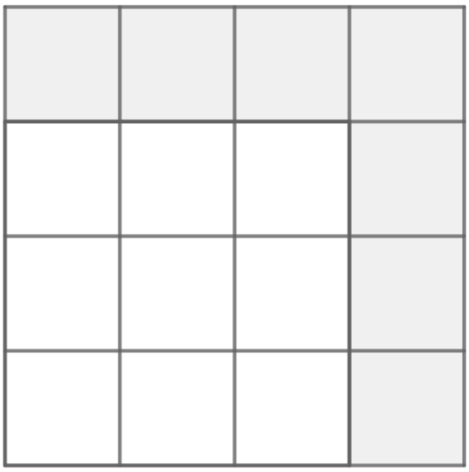
Chapter 4

The Square, S_n





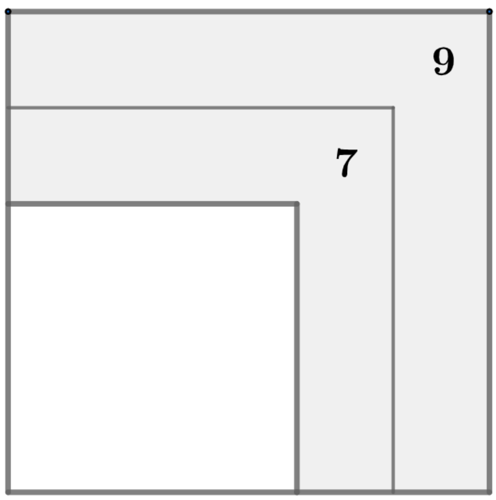
(a) Forming the square, $\boldsymbol{S_n}$

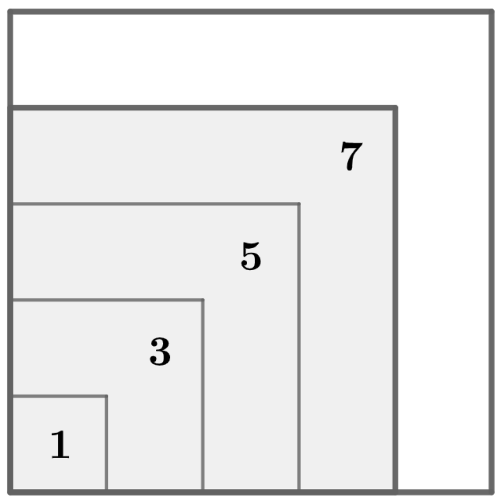


The figure shows that we can characterise the odd number O_n as gnomon to a square, giving the identity

$$\boldsymbol{S_{n-1} + O_n = S_n}. \ \textbf{[4.1]}$$

The sum of the first *n* consecutive odd numbers is therefore the square S_n. But other runs of consecutive odd numbers can also be squares. Where this is the case, the square is that of a term in a Pythagorean triple:

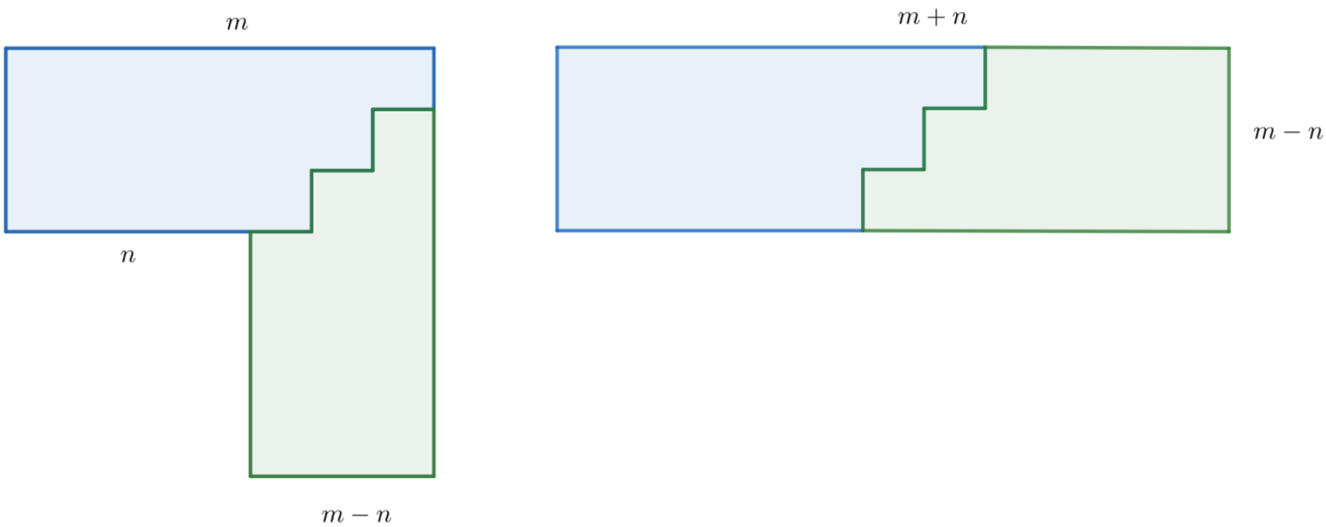




We can show the run of odd numbers comprising the square in two ways by virtue of the commutative law, here as $7 + 9$ or as $1 + 3 + 5 + 7$. (In general, a square might be thus represented in more than two ways. For example, 12^2 can be shown as a run of odd numbers in 5 ways. By contrast, if *p* is prime, one can show that only the sequence starting with 1 can sum to p^2.)

(b) Greek gnomons, $\boldsymbol{GG_{m,n}}$

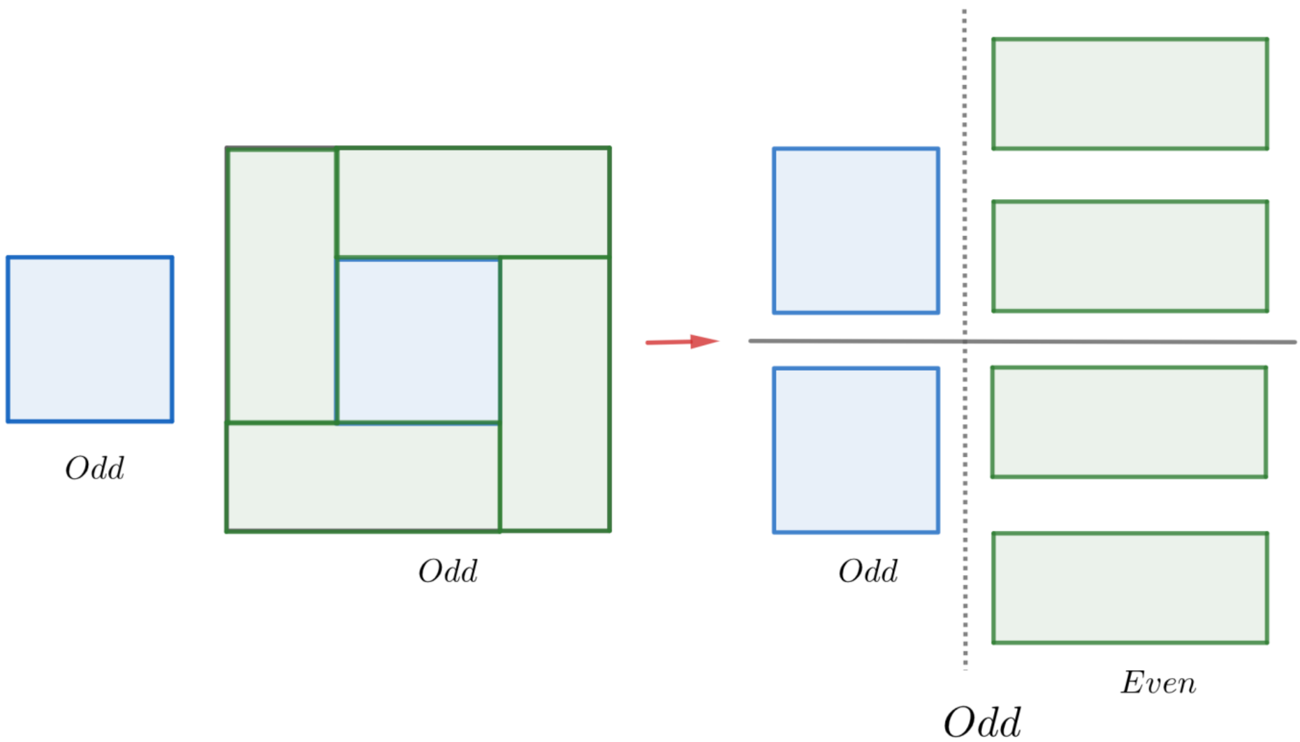
We shall call a sum of one or more consecutive odd numbers a *Greek* gnomon, $GG_{m,n}$. Particular examples (variously arranged) are the (parent) gnomon, grandparent and great grandparent gnomons defined below. As illustrated above, this is a difference of squares, $S_m - S_n = (m + n)(m - n)$, $m > n$. (We include the case $m - n = 1$.) We can also show the Greek gnomon as a rectangle:



The brackets have the same parity. Numbers which are the product of 2 and an odd factor cannot therefore be Greek gnomons. These invalid numbers comprise the arithmetic sequence 2, 6, 10, ..., $2\ O_n$, ... The following algebra shows that this sequence includes all sums of two odd squares:

$$(2a+1)^2+(2b+1)^2$$
$$=4(a^2+b^2+a+b)+2$$
$$=2[2(a^2+b^2+a+b)+1]$$
$$=2(2k+1)\ .$$

Thus two odd squares cannot sum to a Greek gnomon. We can show graphically that the sum of two odd squares takes the form $2(2k+1)$:



$GG_{a+2k,a}$ divides by 4k. This is clear from the algebra: $GG_{a+2k,a} = 4k(a+k)$, also from the fact that the mean of the $2k$ consecutive odd numbers added is an even number, so the total must have factor 4.

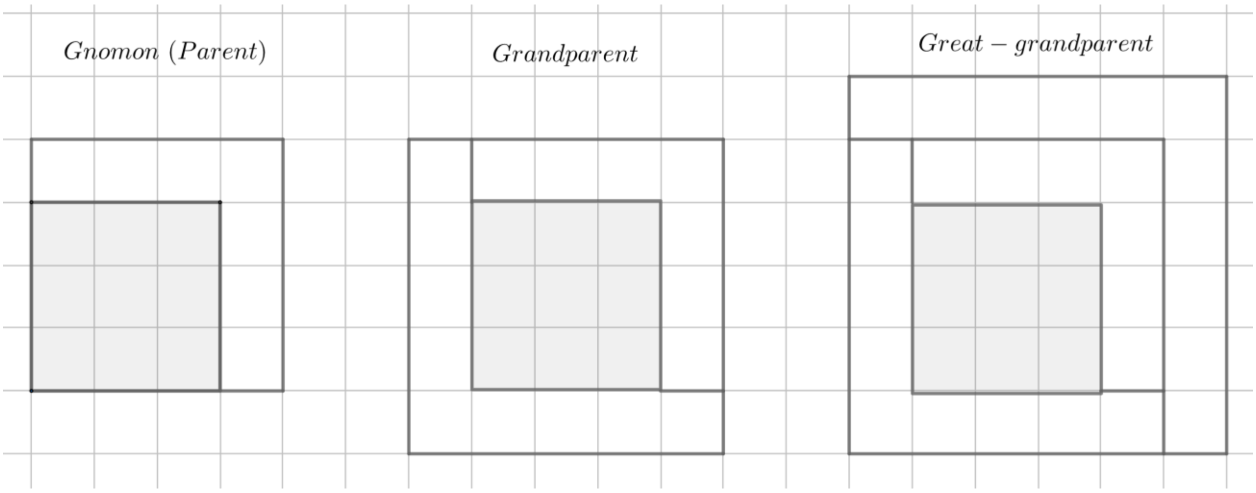
As is also clear from the original figure, we can nest thick gnomons, so that, with $l > m > n$:

$$\boldsymbol{GG_{l,m} + GG_{m,n} = GG_{l,n}}. \textbf{ [4.2]}$$

(c) The gnomon and its relatives

As with the triangle, we can identify grandparents and great grandparents.

For the square, the grandparent can already be drawn to enclose the figure:



The grandparent relation is: $\boldsymbol{S_{n-2} + 4L_{n-1} = S_n}$. **[4.3]**

As in the triangular case, note the rotation symmetry.

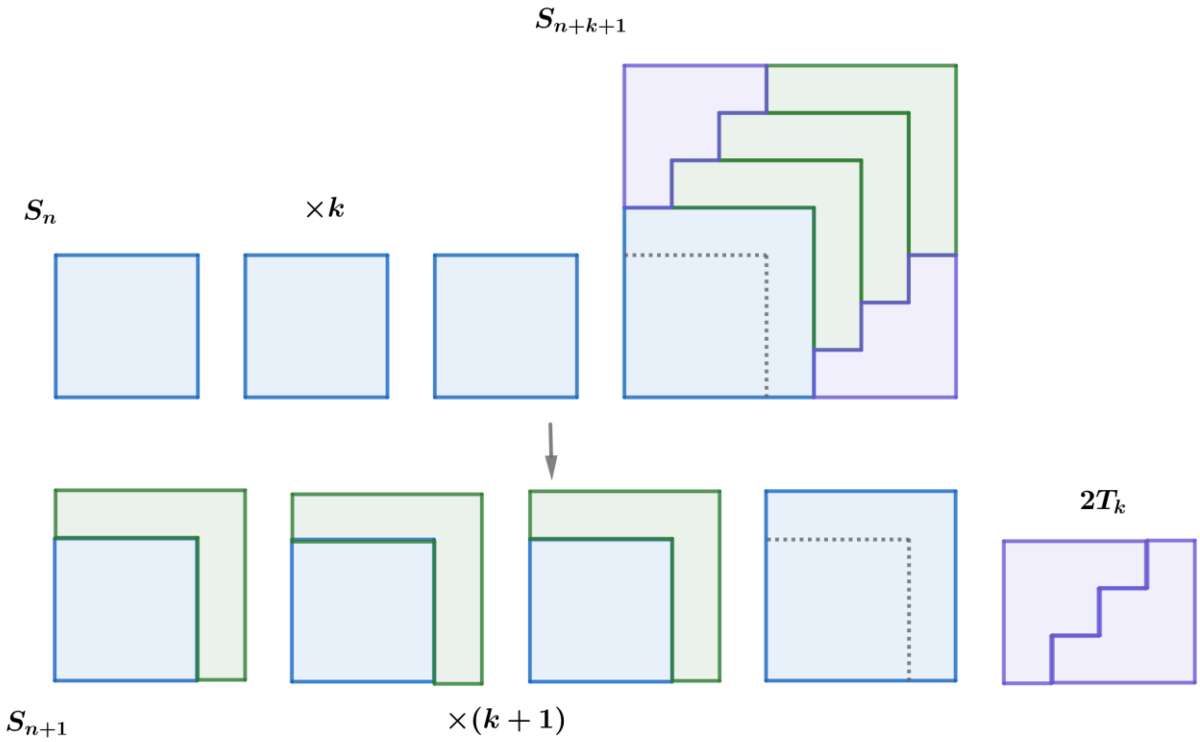
The great-grandparent relation is: $\boldsymbol{S_{n-3} + 3O_{n-1} = S_n}$. **[4.4]**

If we move the central square to the bottom left corner and pack the gnomons from upper right, O_{n-1} appears as the mean of three consecutive odd numbers, whose total therefore is a multiple of 3. (Compare the case of **[3.4]**.)

(d) Nested gnomons

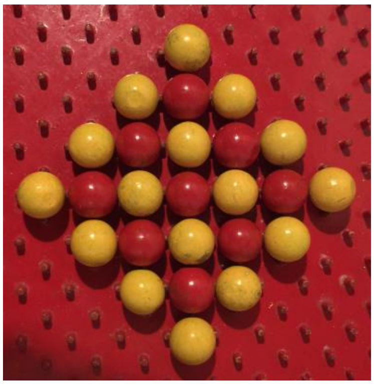
The following dissection represents the identity

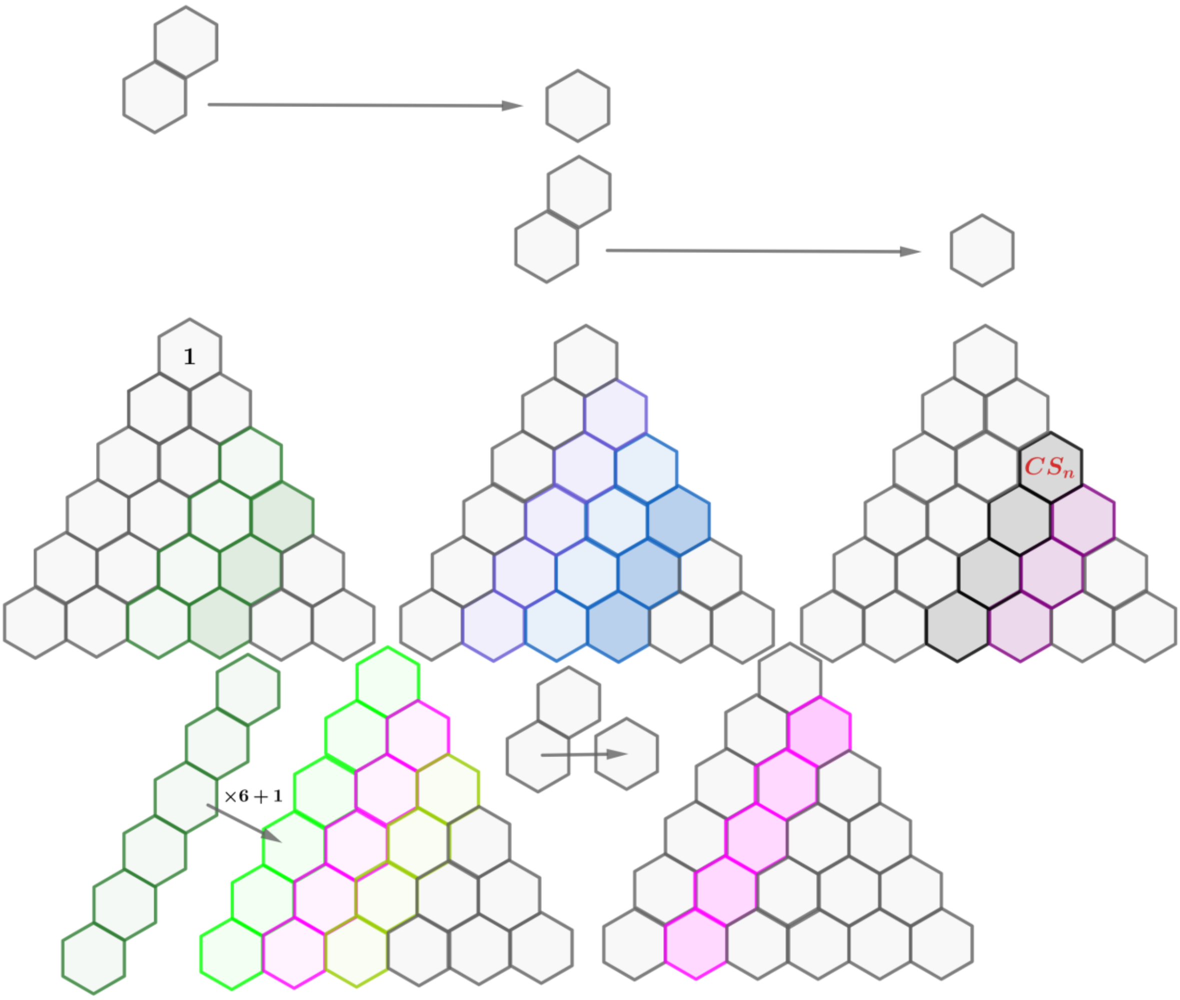
$$\boldsymbol{kS_n + S_{n+k+1} = (k+1)S_{n+1} + 2T_k}. \textbf{ [4.5]}$$


S_{n+k+1}
S_n
$\times k$
$2T_k$
S_{n+1}
$\times(k+1)$

Chapter 5

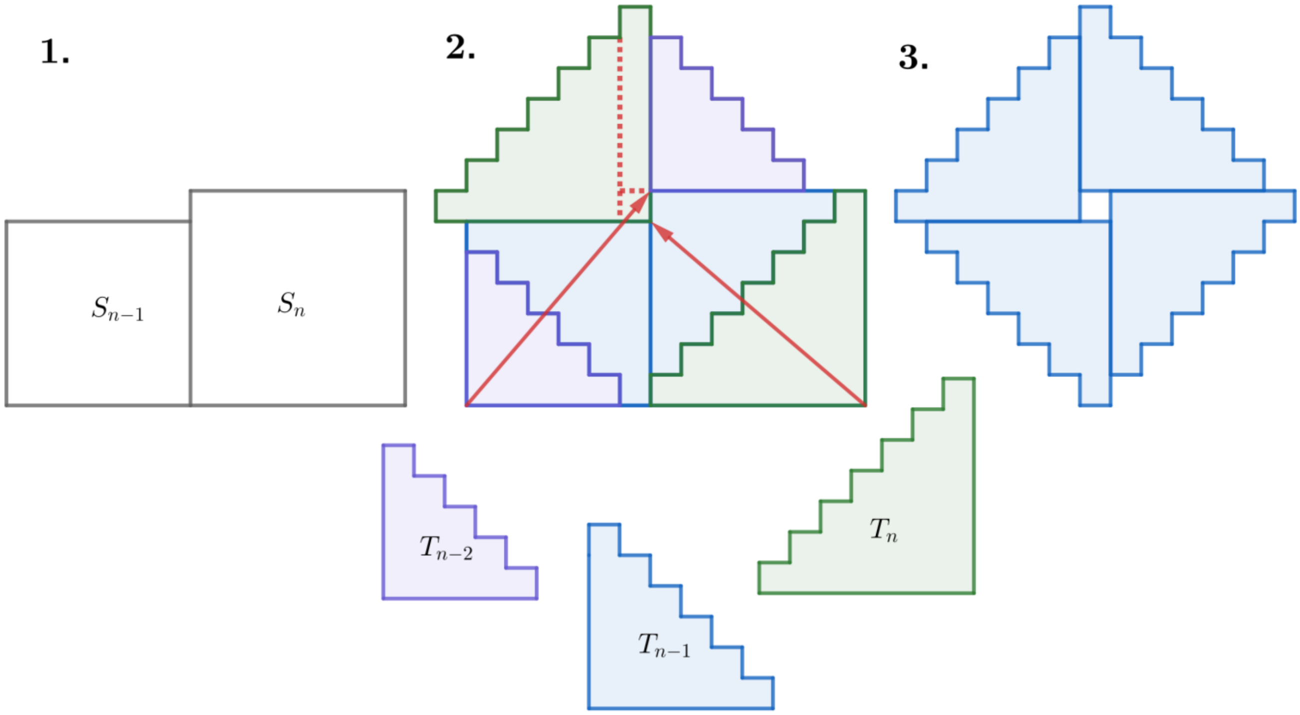
The Centred Square, CS_n





(a) Forming CS_n

The centred square is the sum of two consecutive squares:



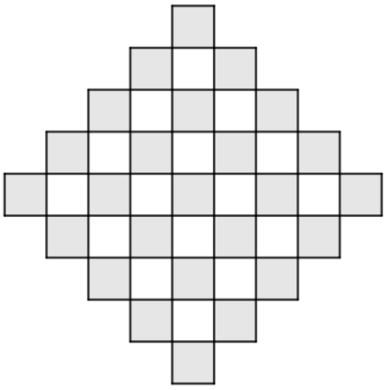
1. We begin with our squares.
2. We divide each into consecutive triangles, so we have:

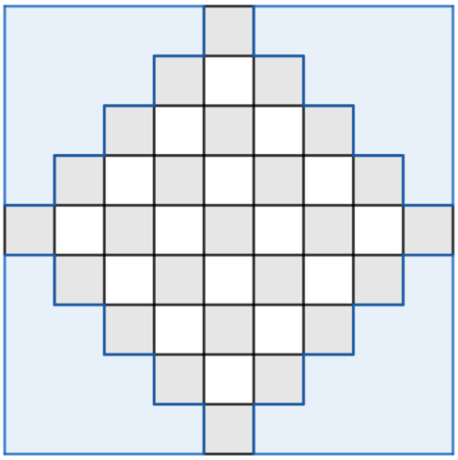
$$S_{n-1} + S_n = T_{n-2} + 2T_{n-1} + T_n. \qquad \textbf{[5.1]}$$

We slide the green triangle bottom right to top left, and the lilac triangle bottom left to top right. We then make the cuts shown by the dotted lines.

3. The net result is the ‘centred square’ number, $CS_n = S_{n-1} + S_n = 4T_{n-1} + 1$. **[5.2]**

The next figure shows part of a tiled floor with a checkerboard pattern. If you turn your head 45° to the vertical, the two consecutive squares reveal themselves. Alongside, we see what happens when we subtract **[2.2]** from **[3.6]** (with change of suffices): $S_{2n-1} - CS_n = 4T_{n-1}$. **[5.3]**





(b) Differences of centred squares

(i) From the defining algebra, we have:

$CS_a - CS_b = 2(a-b)(a+b-1)$.

The brackets have opposite parity, so one of them is positive and the whole expression therefore has a factor 4. This means that, if the right side of the equation is a perfect power, for example a square, the square will be that of an even number.

We can say more.

$2(a-b)(a+b-1) = c^2$
$\Longrightarrow (a+b-1) = 2(a-b)\left(\frac{p}{q}\right)^2$.

Here is a selection of values:

$\frac{p}{q}$	a	b	c
2	5	4	4
1	5	2	6
3	10	9	6
1	11	4	14
$\frac{5}{2}$	14	12	10

The equivalent formula for $CH_a - CH_b = S_c$ differs only in the substitution of the coefficient 3 for 2. The equivalent table starts:

$\frac{p}{q}$	a	b	c
2	7	6	6

$\frac{4}{3}$	10	7	12
$\frac{8}{3}$	34	31	24

(ii) In **Chapter 8**, section **(c) (iii)** we meet Dostor's identity, the second case of which is:

$S_{10} + S_{11} + S_{12} = S_{13} + S_{14}$.

Grouping terms, we have:

$CS_{14} - CS_{11} = S_{12}$,
$CS_{14} - CS_{12} = S_{10}$. (This is the last example in the table above.)

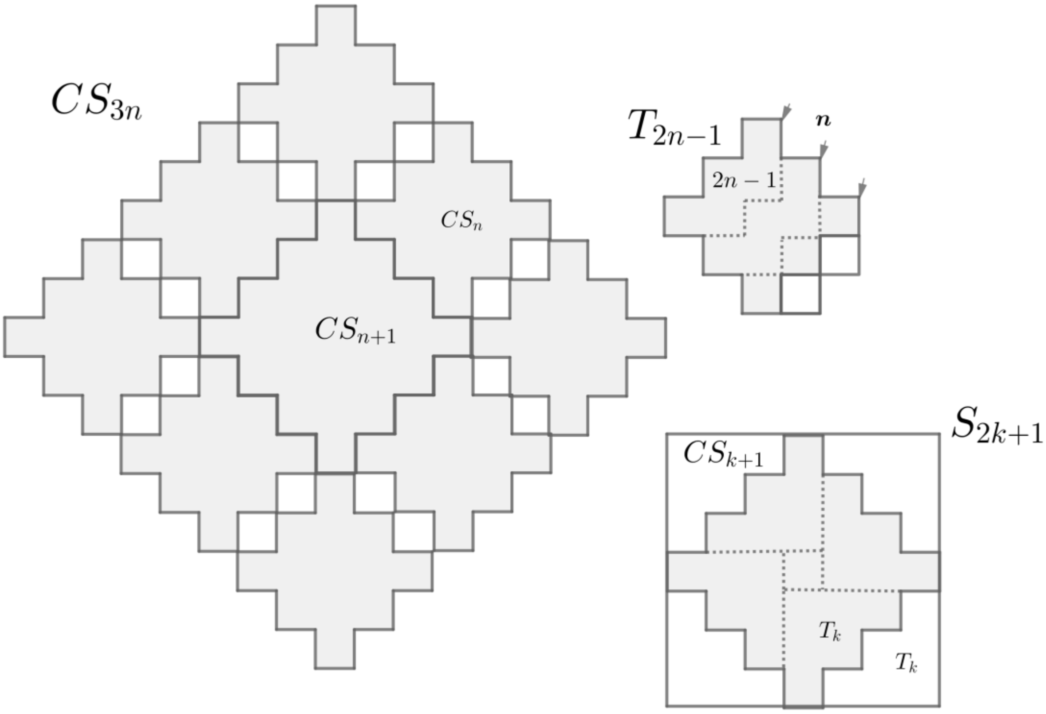
(c) The centred square as a difference of two squares

We write $CS_n = \left(\frac{CS_n+1}{2}\right)^2 - \left(\frac{CS_n-1}{2}\right)^2$.

An instance is $CS_3 = S_7 - S_6$. Rearranging terms, we have $S_2 + S_3 + S_6 = S_7$. The standard house brick has edges in the ratio 2 : 3 : 6. The equation tells us that, with a unit of length so defined, the space diagonal of a brick has integer length, 7.

(d) Identities involving centred squares and other shapes

In the next figure we combine centred squares.



As shown in the main figure, $\boldsymbol{CS_{3n} = CS_{n+1} + 8(CS_n + L_n - 1)}$. **[5.4]**

The upper inset figure shows that $\boldsymbol{CS_n + L_n - 1 = n(2n - 1) = T_{2n-1}}$. **[5.5]**

Combining **[5.4]** and **[5.5]**, $\boldsymbol{CS_{3n} = CS_{n+1} + 8T_{2n-1}}$. **[5.6]**

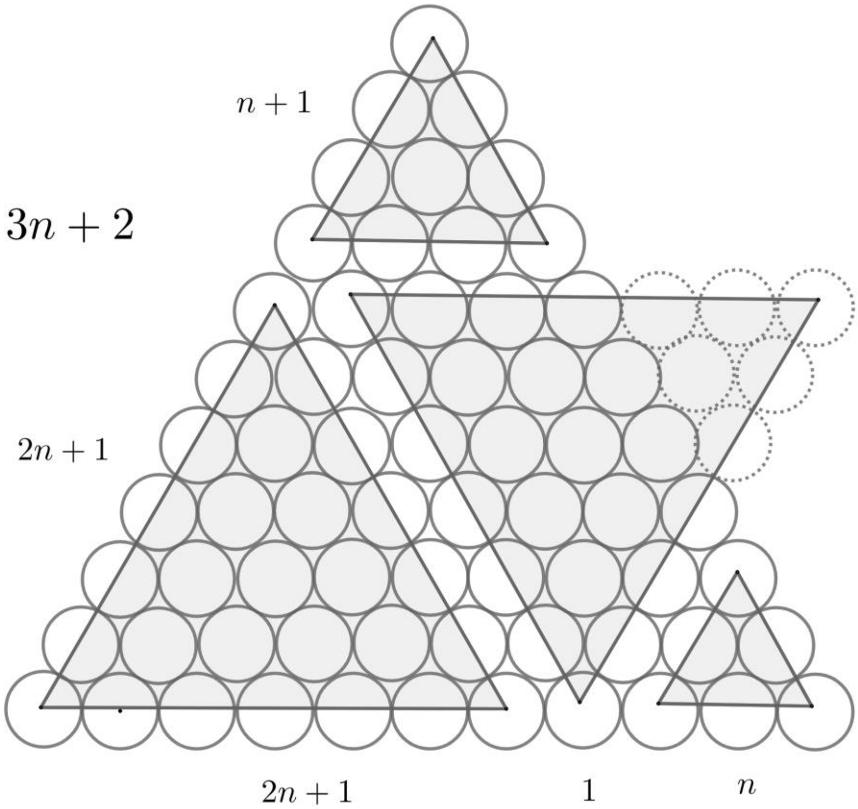
The lower inset figure combines results **[5.2]** and **[5.3]**.

From **[3.6]** and **[5.6]**: $\boldsymbol{CS_{3n} = CS_{n+1} + S_{4n-1} - 1}$. **[5.7]**

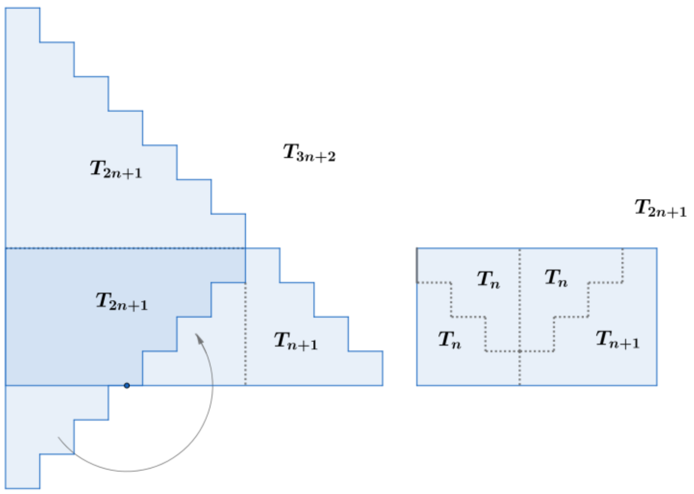
From **[5.2]** and **[5.6]**: $\boldsymbol{T_{3n+2} = T_{n+1} + 2T_{2n+1}}$. **[5.8]**

an identity purely in terms of triangle numbers.

Here is a figure representing **[5.8]**:



Here is the same with a staircase representation. Alongside we use **[3.8]** to dissect the rectangle.

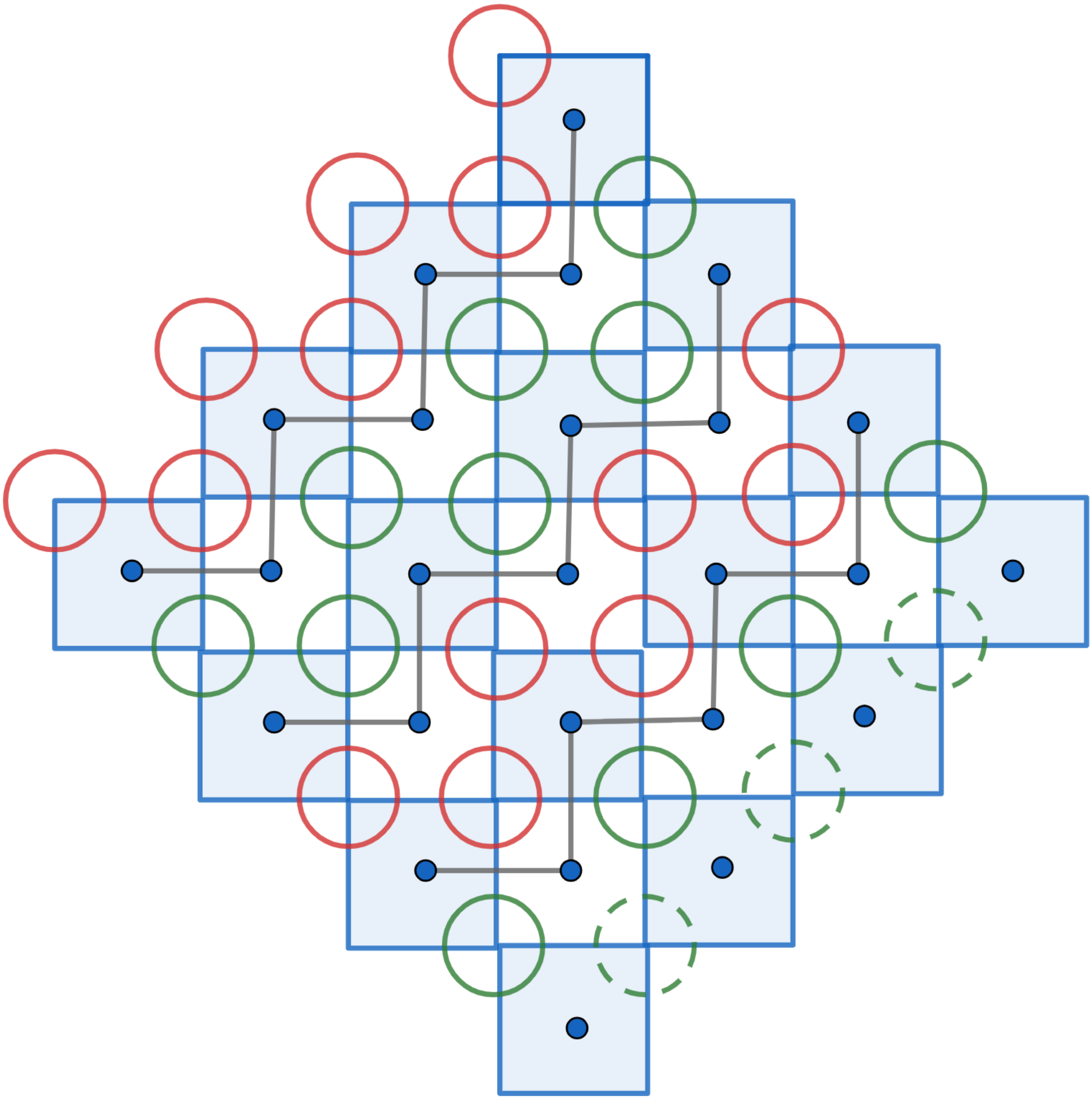


From **[5.2]** we can also break down **[5.6]** into triangles:

$$\boldsymbol{T_{3n+1} + 2T_{3n+2} + T_{3n+3} = 25\,T_n + 10\,T_{n+1} + T_{n+2}}\,. \quad \textbf{[5.9]}$$

From **[3.6]** we have $8T_{n-1} + 1 = S_{2n-1}$,
From [**5.2]** we have $4T_{n-1} + 1 = CS_n$,
Whence $\boldsymbol{2CS_n = S_{2n-1} + 1}$. **[5.10]**

The following figure shows this.



The blue and white squares represent CS_n. Each square is duplicated by a circle top left. The total of squares and circles is therefore $2CS_n$. Each zig-zag line connects $(2n-1)$ objects. Moving top left to bottom right, the $(2n-1)^{th}$ zigzag would require the $(n-1)$ dashed circles. These correspond to n squares on the same diagonal, thereby demonstrating the identity.

(e) The centred square to different moduli

(i) Odd prime moduli

In Chapter 3, section (h) we see how the triangle numbers cycle modulo 3.
Since $S_n = T_{n-1} + T_n$ and $CS_n = S_{n-1} + S_n$, we can find the corresponding cycles for S_n and CS_n by successively adding consecutive pairs of values. We do this for a few prime moduli:

3	T_n :	1 0 0 1 0 0 1 0 0 ...
	S_n :	1 1 0 1 1 0 1 1 0 ...
	CS_n :	1 2 1 1 2 1 1 2 1 ...
5	T_n :	1 3 1 0 0 1 3 1 0 0 1 3 1 0 0 ...
	S_n :	1 4 4 1 0 1 4 4 1 0 1 4 4 1 0 ...
	CS_n :	1 0 3 0 1 1 0 3 0 1 1 0 3 0 1 ...
7	T_n :	1 3 6 3 1 0 0 1 3 6 3 1 0 0 1 3 6 3 1 0 0 ...
	S_n :	1 4 2 2 4 1 0 1 4 2 2 4 1 0 1 4 2 2 4 1 0 ...
	CS_n :	1 5 6 4 6 5 1 1 5 6 4 6 5 1 1 5 6 4 6 5 1 ...

There are two features to note.

First, the length of the cycle is the size of the modulus. To see why this is so, consider how the triangle numbers are formed and the result of working mod 5:

$T_1 = T_0 + 1,$
$T_2 = T_1 + 2,$
$T_3 = T_2 + 3,$
$T_4 = T_3 + 4,$
$T_5 = T_4 + (5),$
$T_6 = T_5 + (5) + 1,$ ← start of new cycle
$T_7 = T_6 + (5) + 2,$
$T_8 = T_7 + (5) + 3,$
...

Second, because of the two 0s at the end of the triangle cycle, the palindromic form of the part before them persists downwards, (as the sequences left of the red lines).

(ii)

Modulo 4

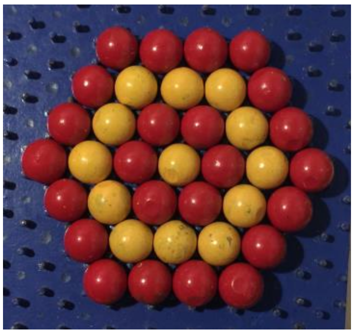
To this modulus, triangle numbers display a cycle of length 8, squares, 2, centred squares, 1:

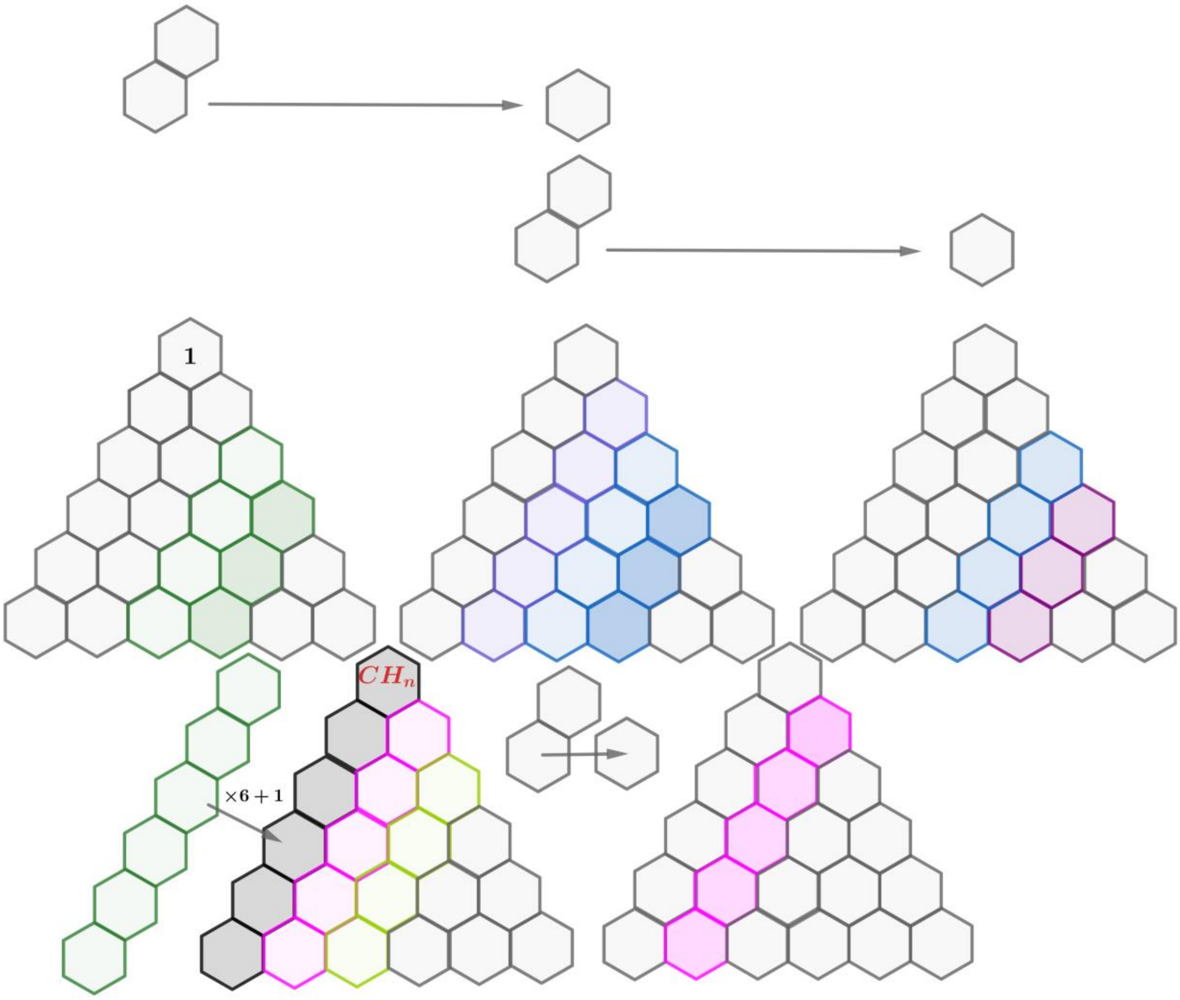
T_n :	1 3 2 2 3 1 0 0 ...
S_n :	1 0 1 0 1 0 1 0 ...
CS_n :	1 1 1 1 1 1 1 1 ...

To other even moduli it is also the case that the cycle length of the triangle numbers is a multiple of those of the squares and centred squares. To modulo 6 the respective lengths are 12, 6, 3; to modulo 8, 16, 4, 4.

Chapter 6

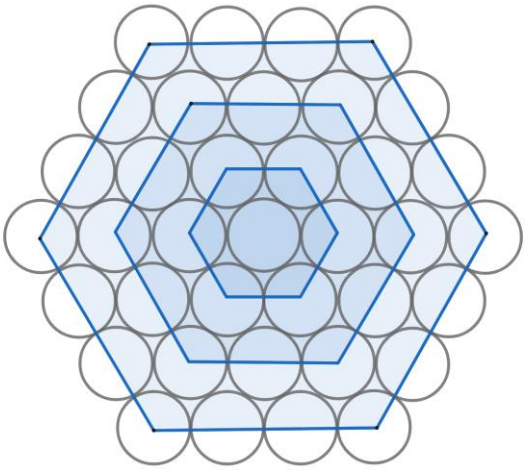
The Centred Hexagon, CH_n



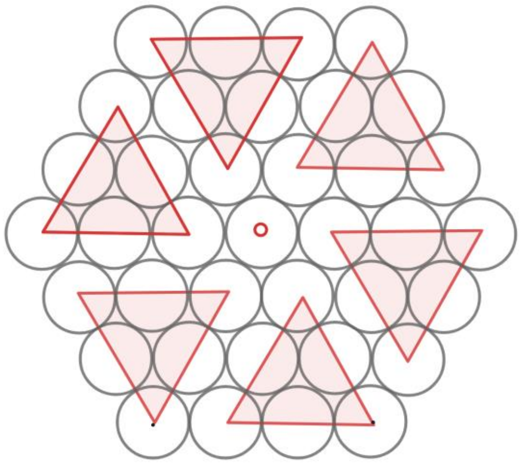


(a) Forming CH_n

The next figure shows how the centred hexagon CH_n is constructed. It includes all the circles on or within the n^{th} hexagonal ring, beginning with the central unit.



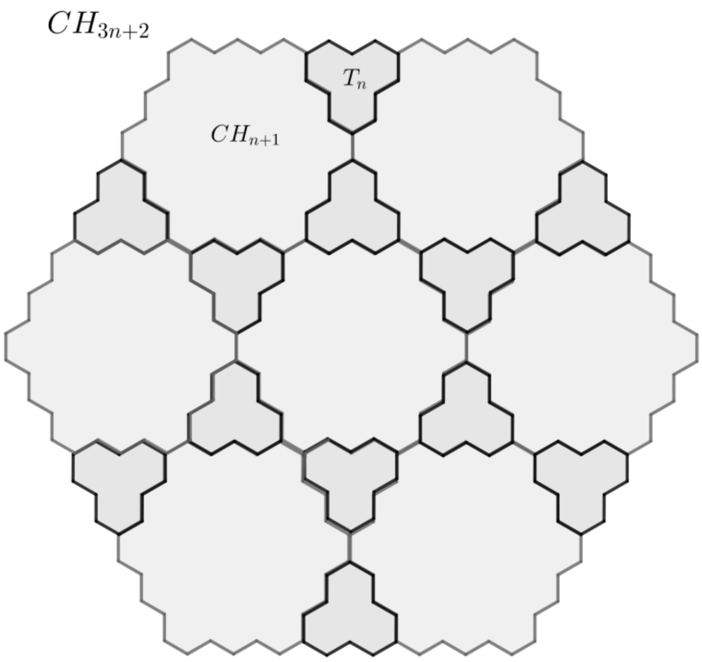
Anticipating **[6.2]**, we see how the common formula can be derived from the figurate formula $6T_{n-1} + 1$.

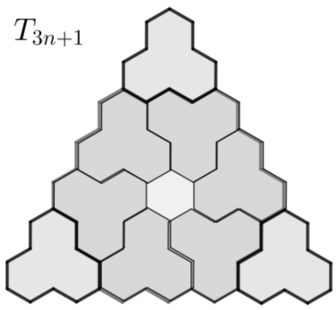


(b) Identities involving centred hexagons and other shapes

In the next figure we combine centred hexagon numbers. Our unit here is a small hexagon.

We have immediately $$\boldsymbol{7CH_{n+1} + 12T_n = CH_{3n+2}}. \quad \textbf{[6.1]}$$





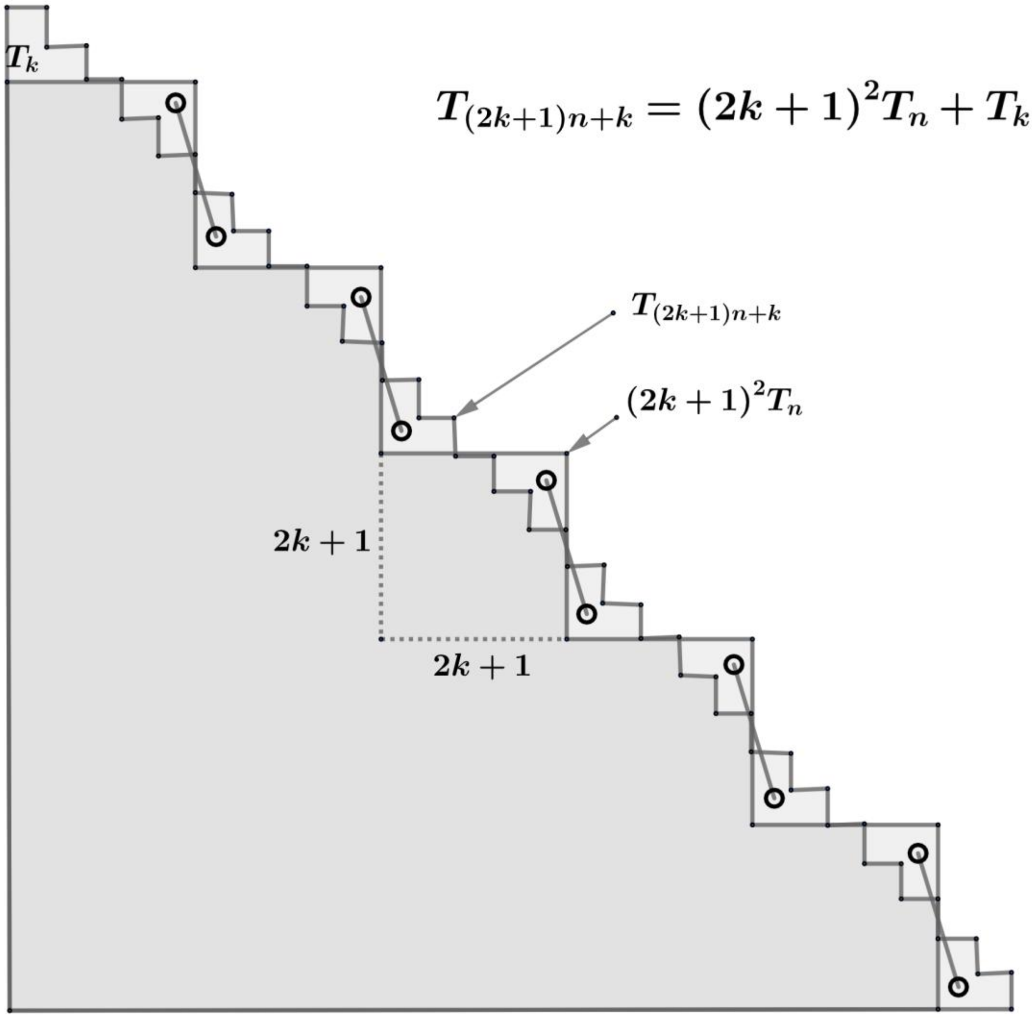
The inset figure, like that above right, gives us $$\boldsymbol{CH_{n+1} = 6T_n + 1}, \quad \textbf{[6.2]}$$
whence $$\boldsymbol{T_{3n+1} = 9T_n + 1}. \quad \textbf{[6.3]}$$

By considering the equation $T_{an+b} = kT_n + l$, we find this result generalises to:

$$\boldsymbol{T_{(2k+1)n+k} = (2k+1)^2 T_n + T_k}. \quad \textbf{[6.4]}$$

(We could write '$2k + 1$' as 'O_{k+1}'.)

We can represent this in figurate terms by making $(2k + 1)^2$ the unit in terms of which we show T_n. We see that the region representing $T_{(2k+1)n+k}$, exceeds the region representing $(2k + 1)^2 T_n$ by T_k:



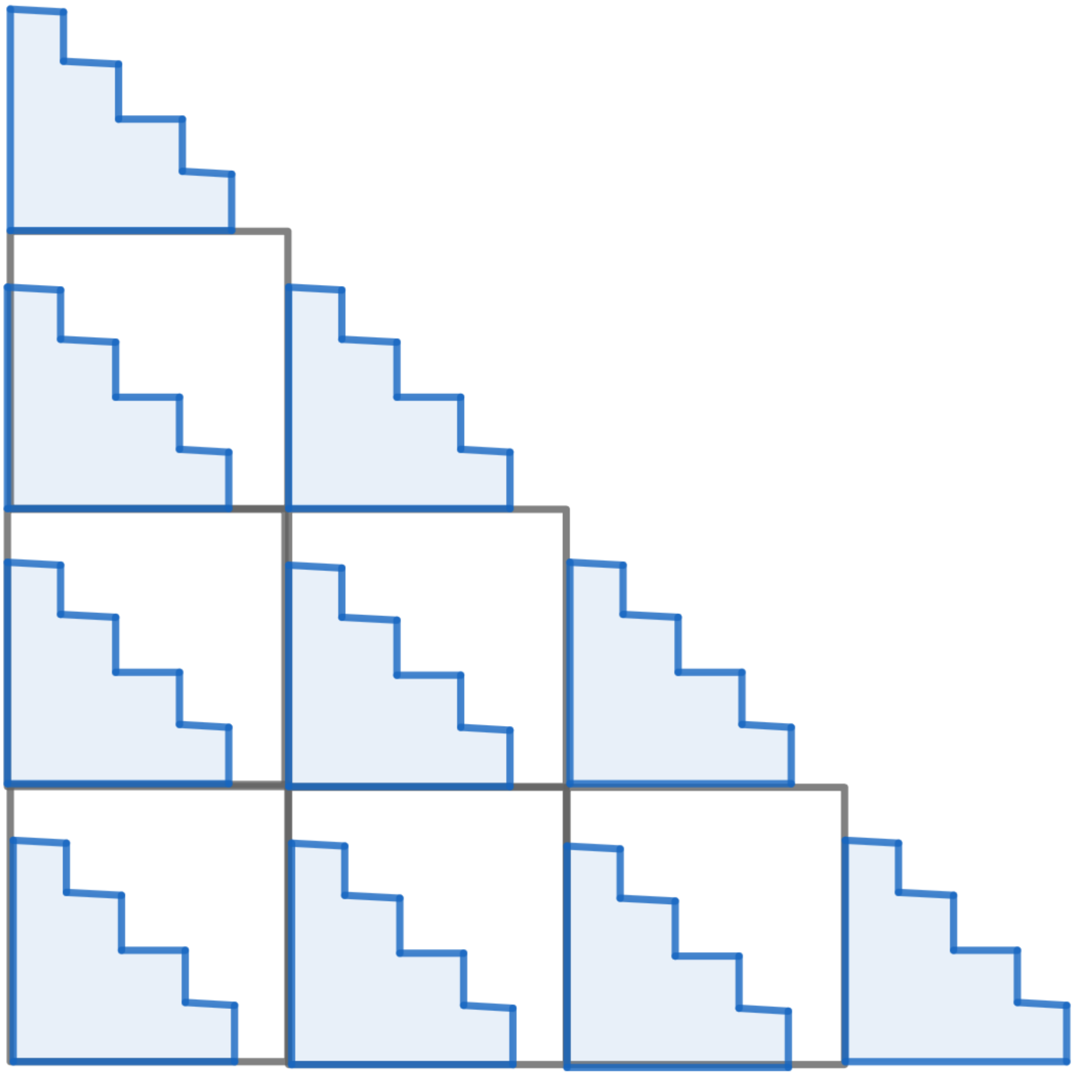
For an alternative dissection see Roger B. Nelsen's own dissection on p. 105 of his book 'Proofs without Words'.

[3.7] and **[5.8]** lead to $\boldsymbol{T_{3n+2} = 3(2T_n + T_{n+1})}$. **[6.5]**

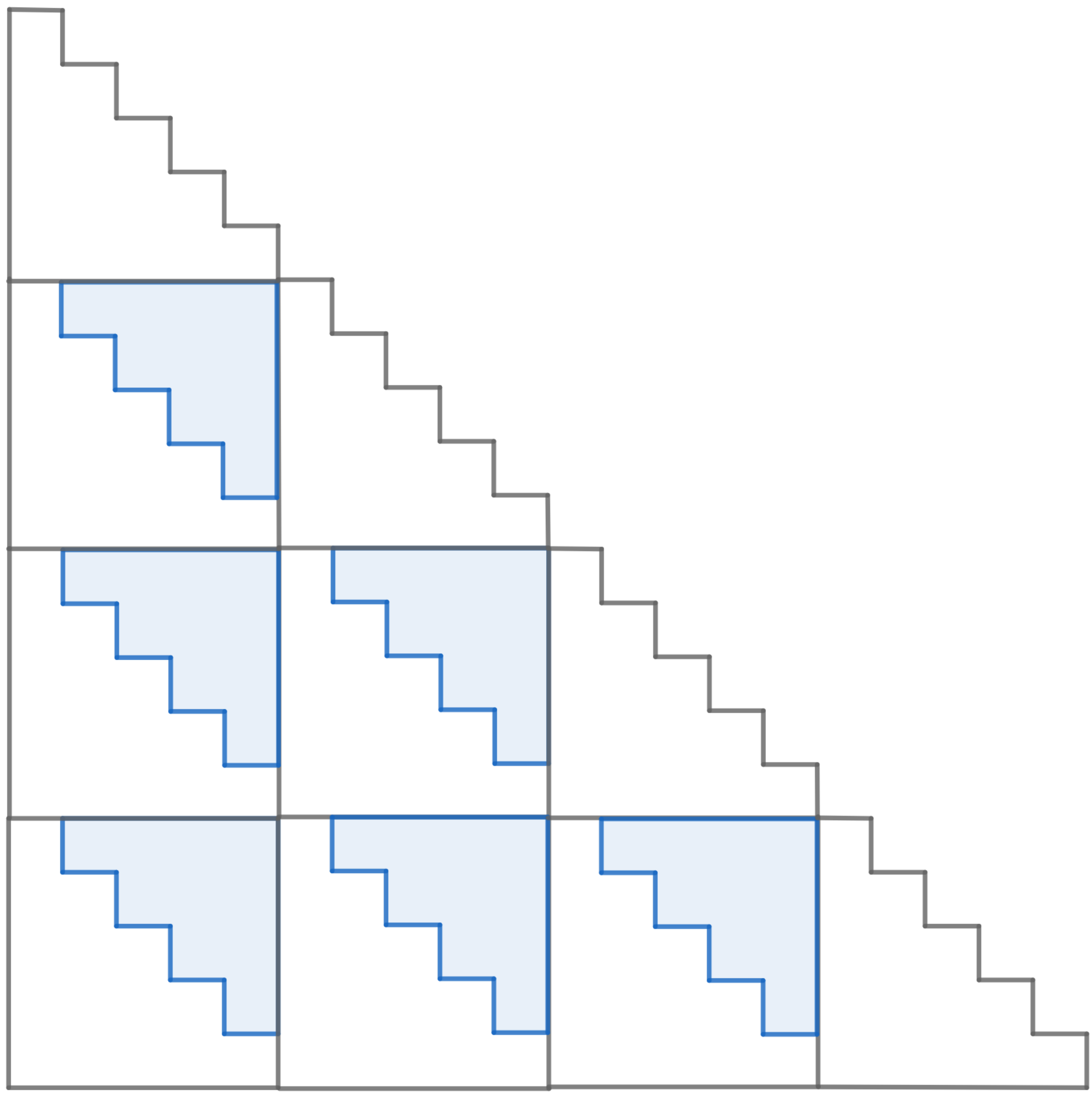
By considering the equation $T_{an+b} = pT_n + qT_{n+1}$, we find this generalises to:

$$\boldsymbol{T_{(k+1)n+k} = T_{(n+1)k+n} = T_{kn+(k+n)} = T_{k+1}T_n + T_k T_{n+1}}. \ \textbf{[6.6]}$$

Note that the common factor in the special case, **[6.5]**, is spurious. But note more significantly the algebraic symmetry. This emerges from the following figure. We can either take T_k to be the number of white triangles, T_{n+1}, and T_{k+1} to be the number of blue triangles, T_n; or T_k to be a blue triangle, of which there are T_{n+1}, and T_{k+1} to be a white triangle, of which there are T_n.



A different identity results if we reverse the size relations so that there are more of the bigger triangle.



This is what Roger B. Nelsen does on p. 99 of his 'Proofs without Words II' to obtain:

$$(\boldsymbol{T_{n-1}})^2 + (\boldsymbol{T_n})^2 = \boldsymbol{T_{S_n}}. \quad \textbf{[6.7]}$$

Nelsen's identity generalises as follows. For one *n* we write *a*; for the other, *b*, producing

$$\boldsymbol{T_{a-1}T_{b-1} + T_aT_b = T_{ab}}. \quad \textbf{[6.8]}$$

We can use our original figure to interpret that. (Again we need more of the bigger triangle.) This time, T_a is the count of triangles of type T_b; T_{a-1} is the count of triangles of type T_{b-1}.

From **[8.1]** we infer $(T_n)^2 - (T_{n-1})^2 = C_n$, whence:

$(T_n)^4 - (T_{n-1})^4$

$= [(T_n)^2 + (T_{n-1})^2][(T_n)^2 - (T_{n-1})^2]$
$= T_{S_n} C_n,$ $\qquad$ $\boldsymbol{(T_n)^4 - (T_{n-1})^4 = T_{S_n} C_n}$. **[6.9]**

If *n* is odd, $T_{S_n} = \frac{S_n(S_n+1)}{2}$ has a factor S_n, so $T_{S_n} C_n$ has a factor $(L_n)^5$; if *n* is odd, a factor $(L_n)^4$.

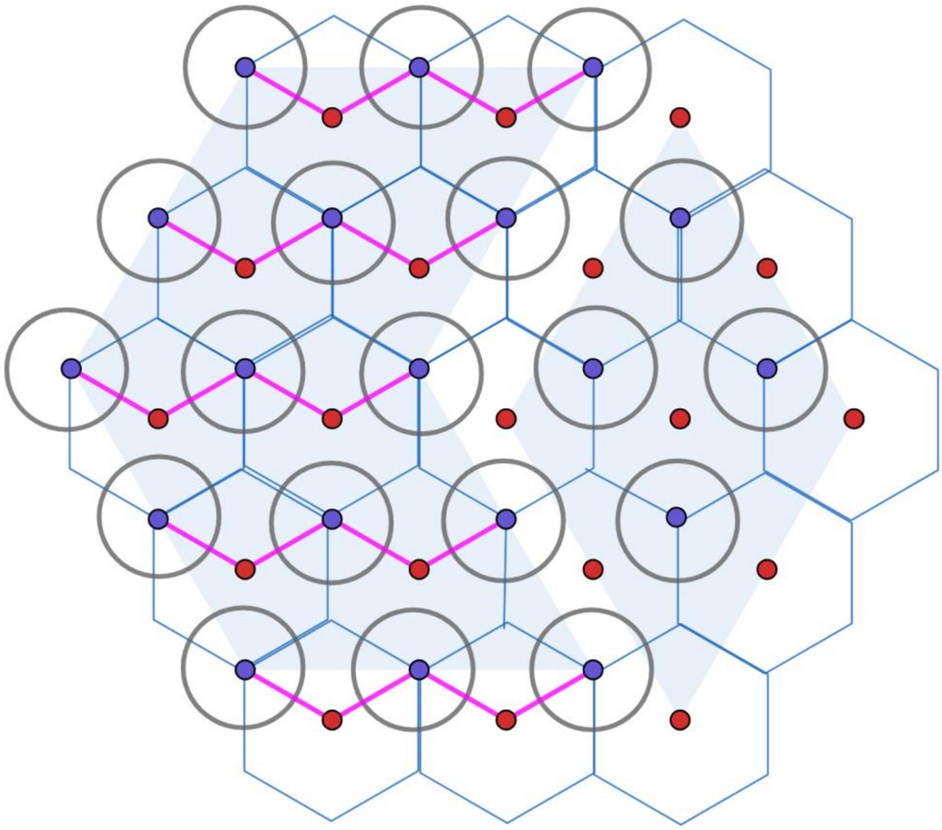
Returning to **[6.3]** and substituting from **[3.6]**, we have $\boldsymbol{T_{3n+1} = S_{2n+1} + T_n}$. **[6.10]**

From **(5.2)** and **(6.2)** we derive $\boldsymbol{CH_n - CS_n = 2T_{n-1}}$. **[6.11]**
Recalling the first two of the following identities and adding the last:

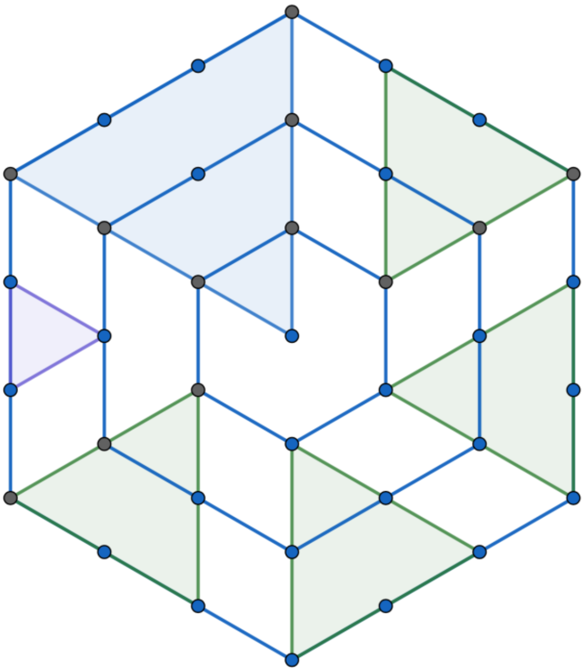
$CS_n = 2n(n-1) + 1,$
$CH_n = 3n(n-1) + 1,$
$S_{2n-1} = 4n(n-1) + 1,$

we have: $\boldsymbol{CH_n = \frac{CS_n + S_{2n-1}}{2}}$. **[6.12]**

The following figure illustrates this. We have shown CH_n in terms of hexagons. Each hexagon has a red dot at the centre. A circle with a mauve centre is centred on the upper left vertex of each hexagon. The circles thus duplicate the hexagons and the number of circles plus hexagons, red dots plus mauve dots, is $2CH_n$. We partition these between the chevron on the left, enclosing S_{2n-1} dots and the rhombus on the right, enclosing CS_n dots.



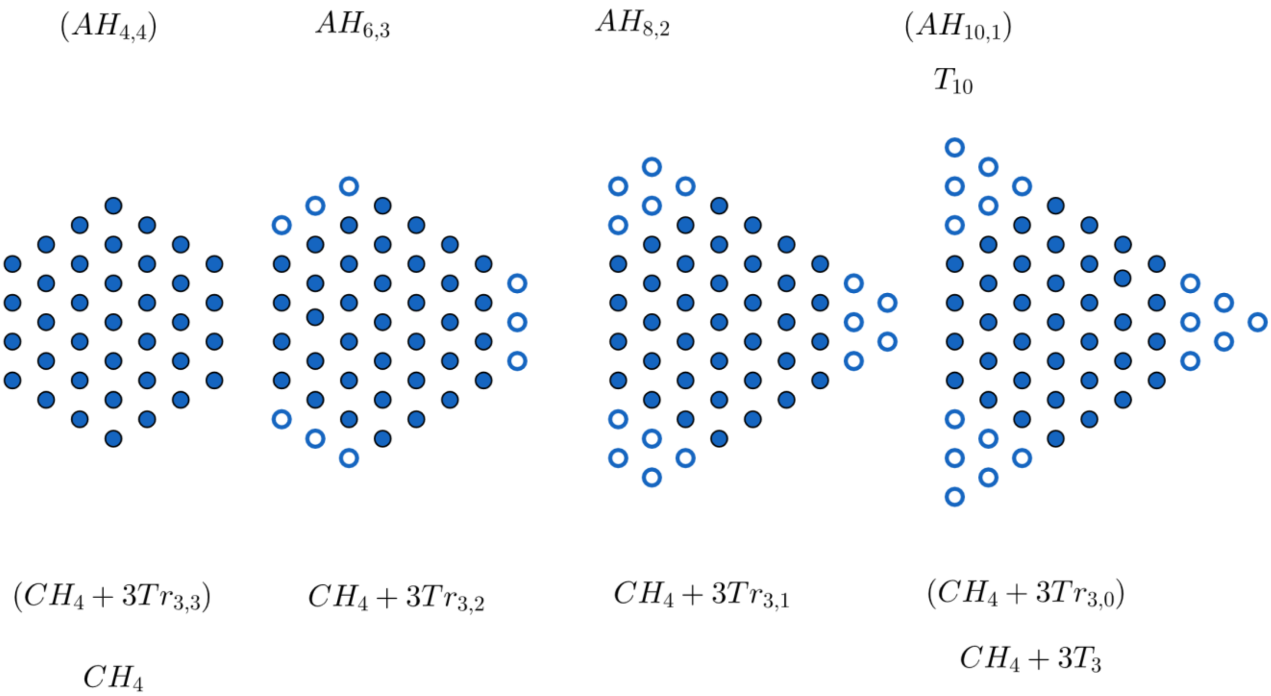
The dissection below shows: $\boldsymbol{CH_n = T_{n-2} + 4T_{n-1} + T_n}$. **[6.13]**



When we come to consider CH numbers as gnomons to a cube, we shall meet the 3-D analogue of this identity.

The alternate hexagon, $AH_{m,n}$

We can derive from the centred hexagon, whose sides have equal length, an equiangular hexagon whose sides are of alternate lengths m, n. We shall call this the alternate hexagon, symbol $AH_{m,n}$, $m > n > 0$. The following figure shows the sequence developed from CH_4. The symbol sequences include limiting cases in order to illustrate the progressions.



Algebra confirms the identity instanced here: $\boldsymbol{T_{3n-1} = CH_n + 3T_{n-1}}$. **[6.14]**

By generalising the expressions in centred hexagons and trapezoids, moving left to right as it were, we find:

$AH_{m,n} = CH_{\frac{m+2n}{3}} + 3\left[T_{\frac{m+2n-3}{3}} - T_{n-1}\right]$, an unnecessarily complex expression.

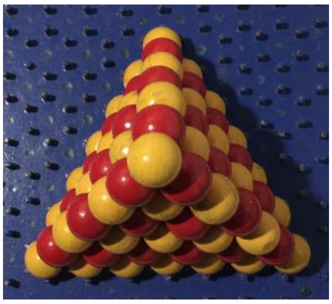
If we move right to left, we can begin with a triangle and truncate it progressively. Working in terms of triangles rather than centred hexagons and trapezoids, we derive the simpler form:

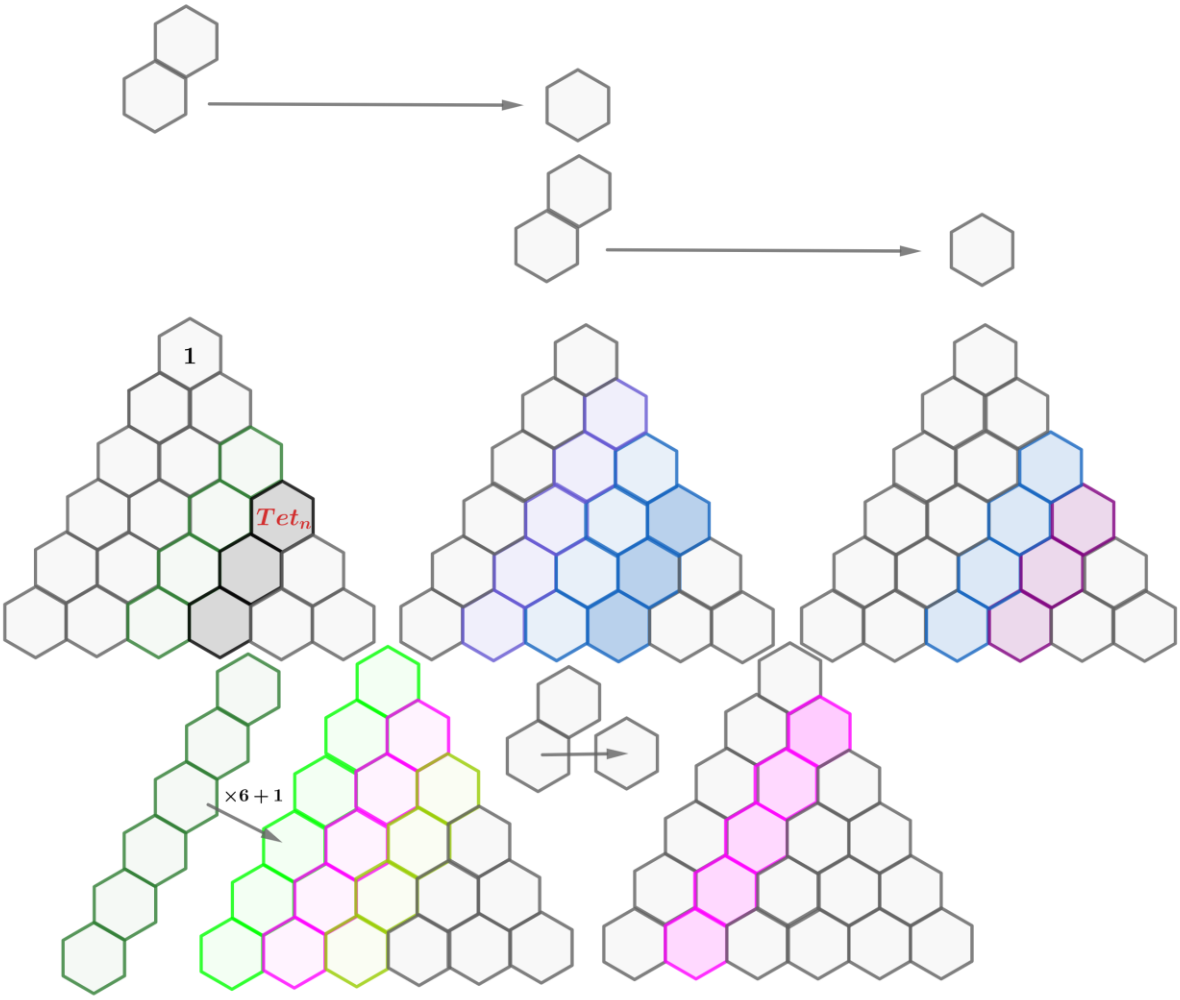
$AH_{m,n} = T_{m+2(n-1)} - 3T_{n-1}$.

As with other 2-parameter shapes, two different symbols can code the same number, For example, $AH_{5,3} = T_9 - 3T_2 = AH_{8,1} = T_8$.

Chapter 7

The Tetrahedron, Tet_n





(a) Forming Tet_n

To make a tetrahedron, we stack triangles. T_n is gnomon to Tet_{n-1} as, in two dimensions, n is gnomon to T_n .

$$\boldsymbol{Tet_{n-1} + T_n = Tet_n} \text{ . [7.1]}$$

Alternatively, we may section the packing in rectangular slabs:

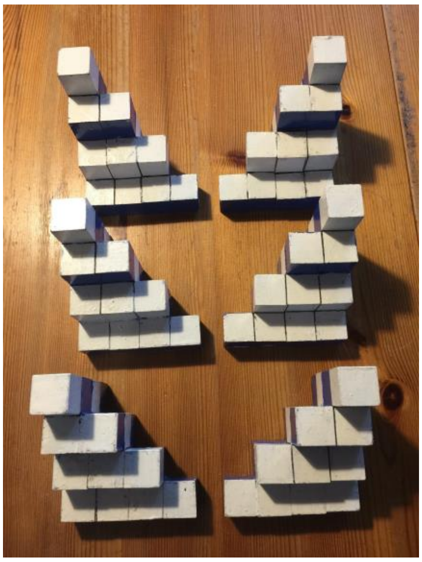
On the multiplication square these appear on a line perpendicular to the main diagonal since the orange numbers are symmetrical about it.

This dissection corresponds to the sum $Tet_n = \sum_{i=1}^{i=n} i(n+1-i)$.

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135
10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165
12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195
14	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210
15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225

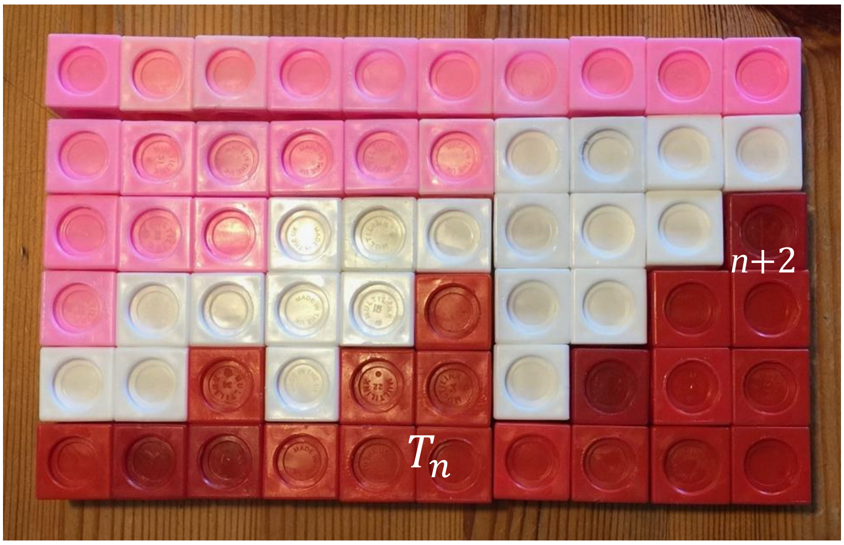
(b) Its algebraic formula

A dissection demonstrating the general formula is found on p. 95 of Roger B. Nelsen's 'Proofs without Words'. There are two chiral sets of 3 congruent shapes, which fit together to make a congruent pair of staircases. Notice how chiral pairs match.



We fold one staircase on top of the other to make a cuboid having dimensions n , $n+1$, $n+2$. The volume of the n^{th} tetrahedron is then 1/6 of this product.

For a two-dimensional dissection, see that of Monte J. Zerger on p. 94 of the work cited above:



$$3Tet_n = T_n \times (n+2),$$
$$Tet_n = T_n \times \frac{n+2}{3}$$
$$= \frac{n(n+1)}{2} \times \frac{n+2}{3}$$
$$= \frac{n(n+1)(n+2)}{6}.$$

(c) Gnomonic relations

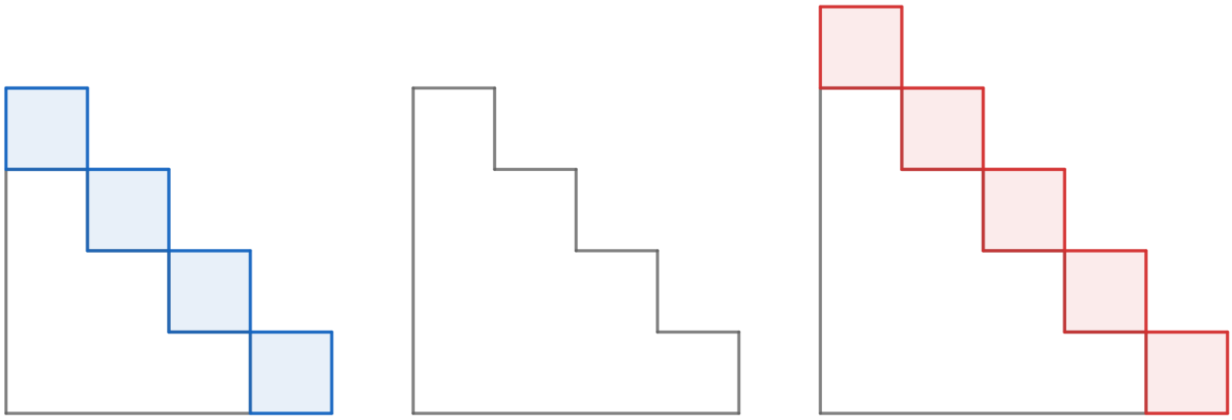
We can also have gnomonic grandparents as we did in two dimensions.
The same figure which illustrated the two-dimensional case serves to distinguish the two possible arrangements here, this time of spheres rather than disks.

The resulting identity is $\boldsymbol{Tet_{n-2} + (T_{n-1} + T_n) = Tet_{n-2} + S_n = Tet_n}$. **[7.2]**

As in two dimensions, we can draw the great-grandparent enclosing the original figure. The identity here is:

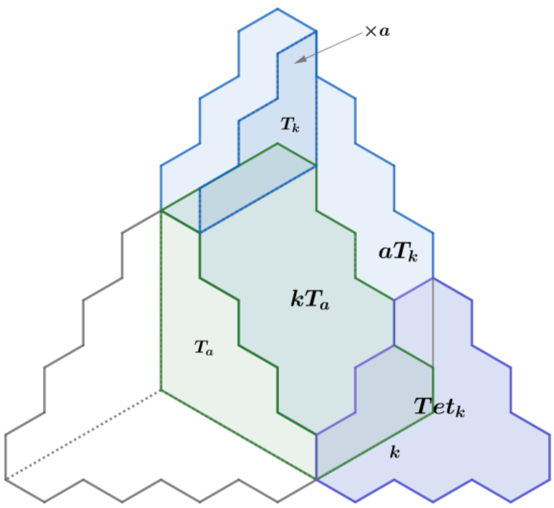
$$\boldsymbol{Tet_{n-3} + (T_{n-2} + T_{n-1} + T_n) = Tet_{n-3} + 3T_{n-1} + 1 = Tet_n} \text{ . } \mathbf{[7.3]}$$

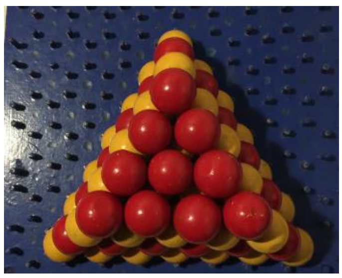
We have used the identity $T_{n-2} + T_{n-1} + T_n = 3T_{n-1} + 1$, which is **[3.7]** in disguise, but may be shown in this form by the following figure, where we see that the red cells exceed the blue cells by 1. **[3.7]** emerges from omitting the middle triangle.



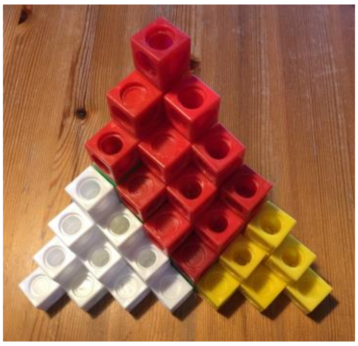
(d) The difference of two tetrahedra (the frustum of a tetrahedron, the general gnomon to a tetrahedron)

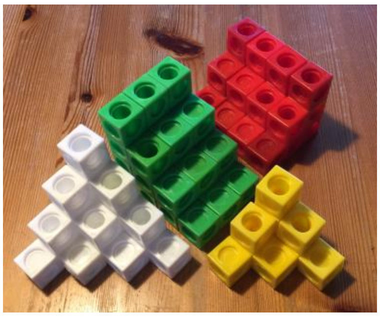
$Tet_{a+k} - Tet_a$ is the sum of the k consecutive triangles, beginning with T_{a+1} .
The algebra yields this number as $\frac{k}{6}[3(a^2 + ak + 2a + k) + (k^2 + 2)]$ or
or $kT_a + aT_k + Tet_k$. Here is the frustum thus dissected:

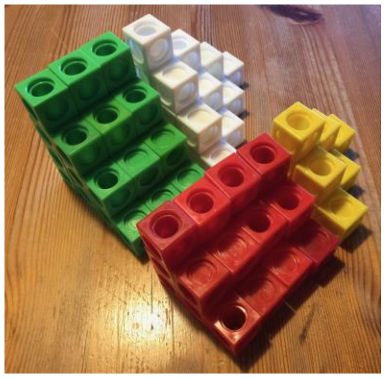




Note that, if we add back the empty piece, which is Tet_a, to complete the tetrahedron Tet_{a+k}, we have, as required, an expression symmetrical in a and k: $Tet_{a+k} = aT_k + kT_a + aT_k + kT_a$. The following pictures bring out these relations. Note how the a triangular slabs, T_k , laid along the steps of the green staircase, kT_a, constitute a staircase of their own, aT_k.







(e) The divisibility of the sum of *k* consecutive triangles

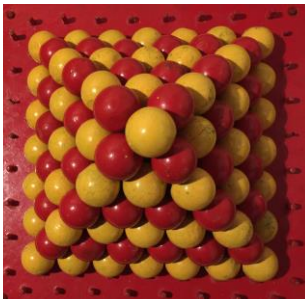
(i) Inspecting the first expression in **(d)**, we see that, if k is a prime > 3, since it shares no factors with 6, it must survive cancelling of the whole expression and therefore be a divisor of all sums of nk consecutive squares.

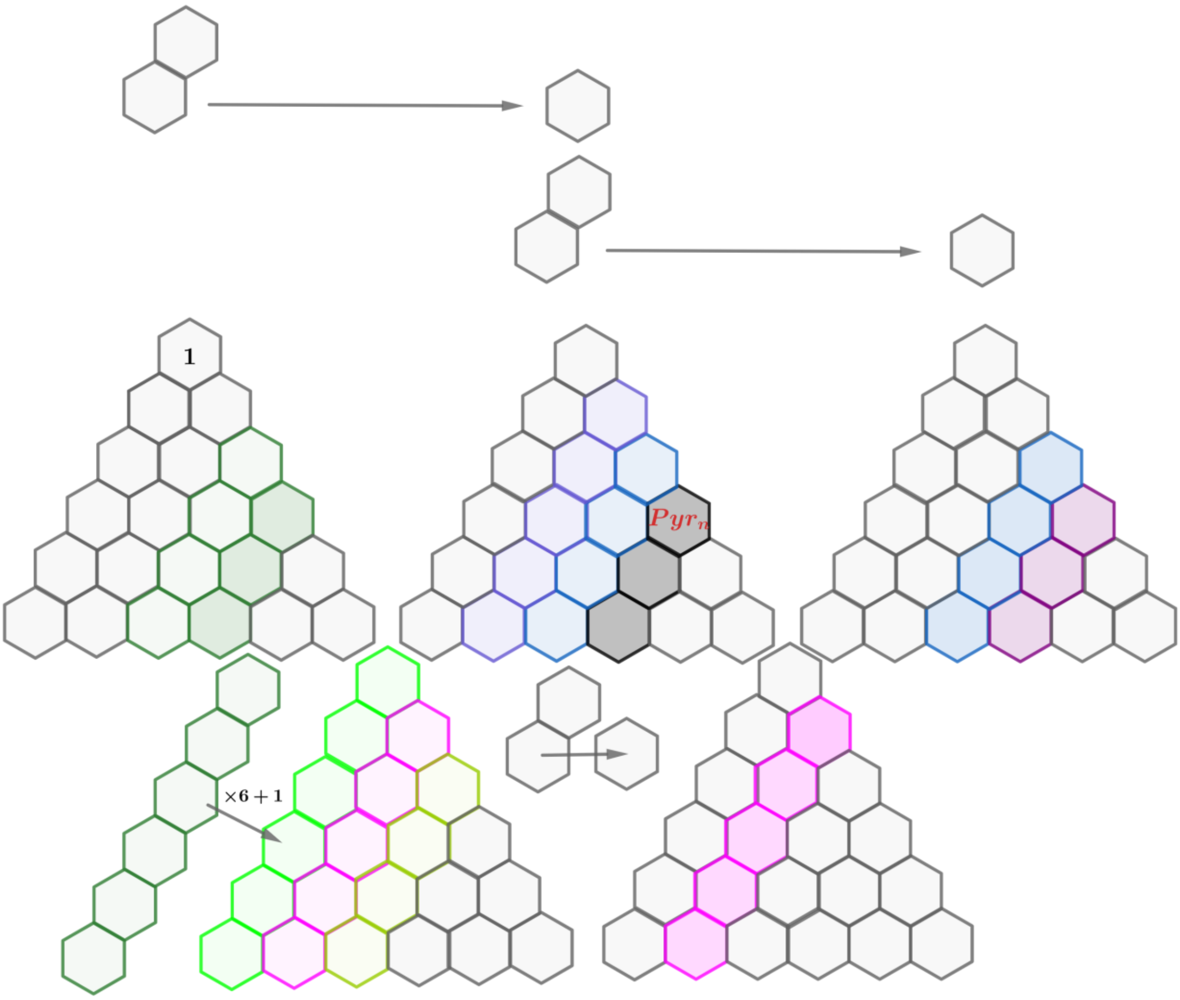
(ii) If k contains the factors 2^a or 3^b, $a, b > 1$, the exponents a, b will fall by at most 1 on cancellation with the '6' in the denominator, and so a divisor of the total will retain the factors 2^{a-1} or 3^{b-1}.

(iii) Combining **(i)** and **(ii)**, we conclude that, for p a prime > 3, a, b >1,
$2^{a-1}3^{b-1}p$ divides the sum of $2^a 3^b np$ consecutive triangles.

Chapter 8

The Pyramid, Pyr_n



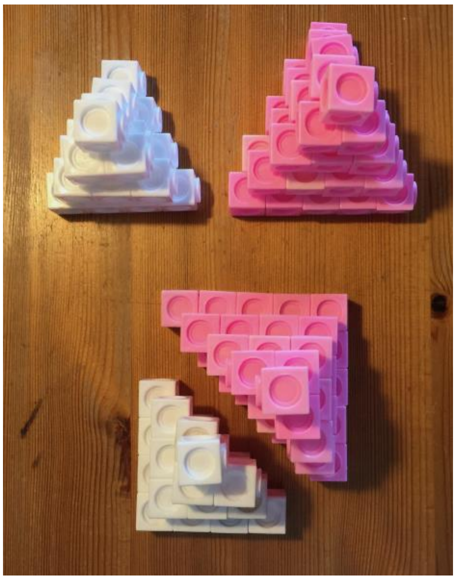


(a) Forming $\boldsymbol{Pyr_n}$

To make a pyramid, we stack squares:

$$\boldsymbol{Pyr_{n-1} + S_n = Pyr_n}. \quad \textbf{[8.1]}$$

These are shown in yellow on the multiplication square below.



We can compare the figures layer by layer.

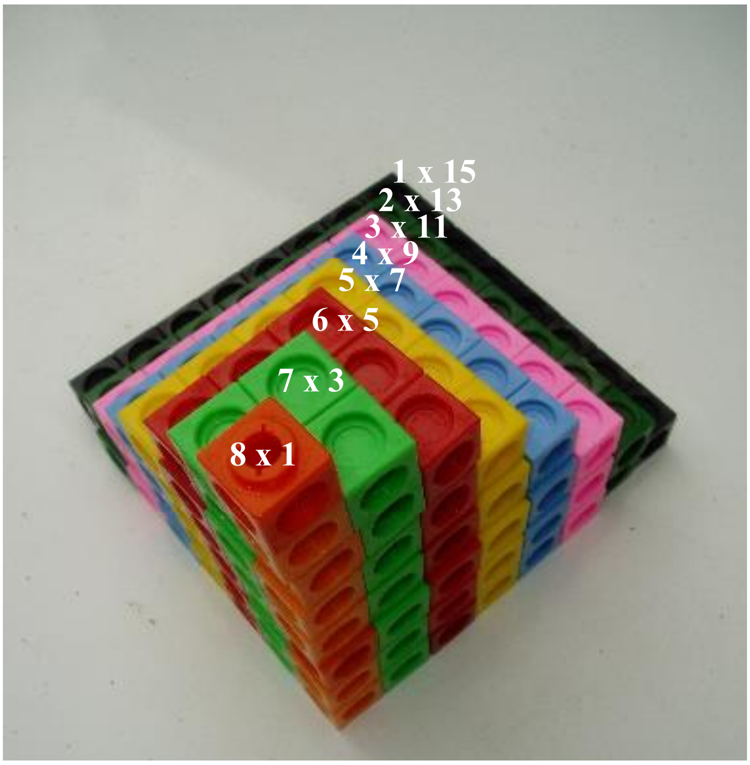
As we had $T_{n-1} + T_n = S_n$, we have

$$\boldsymbol{Tet_{n-1} + Tet_n = Pyr_n}. \quad \textbf{[8.2]}$$

In terms of the multiplication square, and our dissection of the tetrahedron in rectangular slabs, **[5.2]** requires us to add the cells shown in orange and blue beneath. The column totals appear in the green cells. These also result from a different dissection of the pyramid. In the colour-coded picture below we have taken a corner pyramid and divided it into L-shaped prisms. This dissection corresponds to the sum

$$Pyr_n = \sum_{i=1}^{i=n}(2i-1)(n+1-i).$$

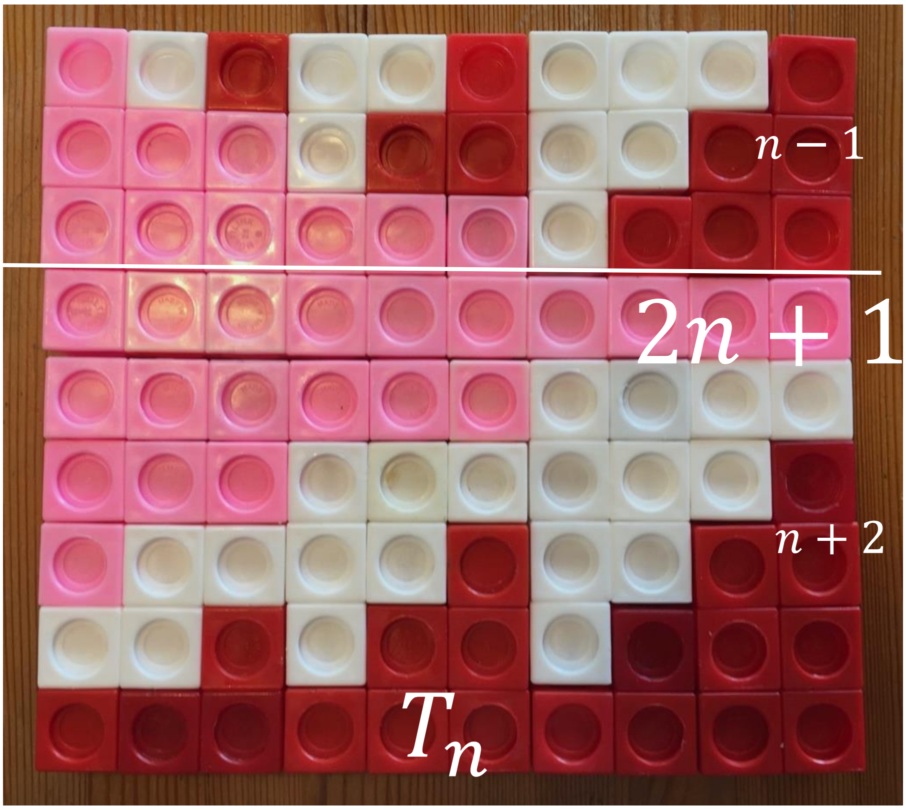
Compare this expression with the corresponding one for Tet_n.



X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135
10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165
12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195
14	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210
15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225

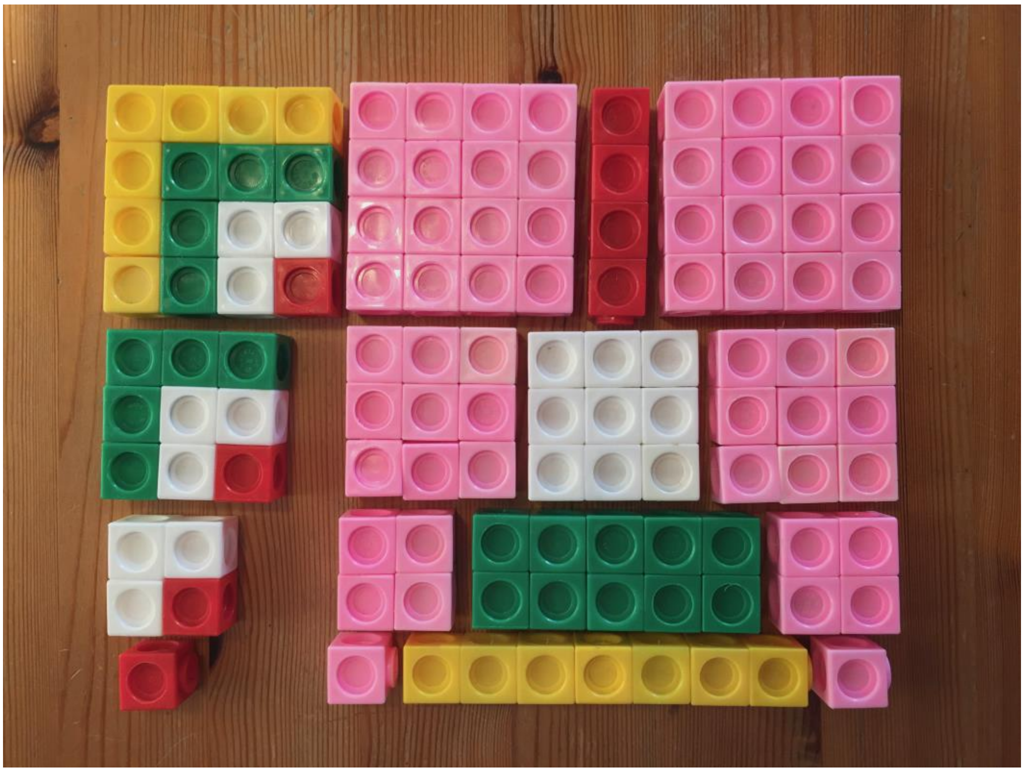
The general formula emerges from this dissection based on that by Man-Keung Siu, found on p. 77 of Roger B. Nelsen's 'Proofs without Words'. There are 6 congruent pieces, fitted together in threes to make two congruent blocks, which in turn fit together to make a cuboid with dimensions $n, n+1, 2n+1$. The volume of the n^{th} pyramid is then 1/6 of the product.

Monte J. Zerger's dissection exhibiting the formula for Tet_n can be adapted to show Pyr_n as follows.



$$3Pyr_n = T_n \times (2n+1),$$
$$Pyr_n = T_n \times \frac{2n+1}{3}$$
$$= \frac{n(n+1)(2n+1)}{6}.$$

Here is an alternative dissection. This is due to Dan Kalman and Martin Gardner and is found on p. 78 of Roger B. Nelsen's 'Proofs without Words'.

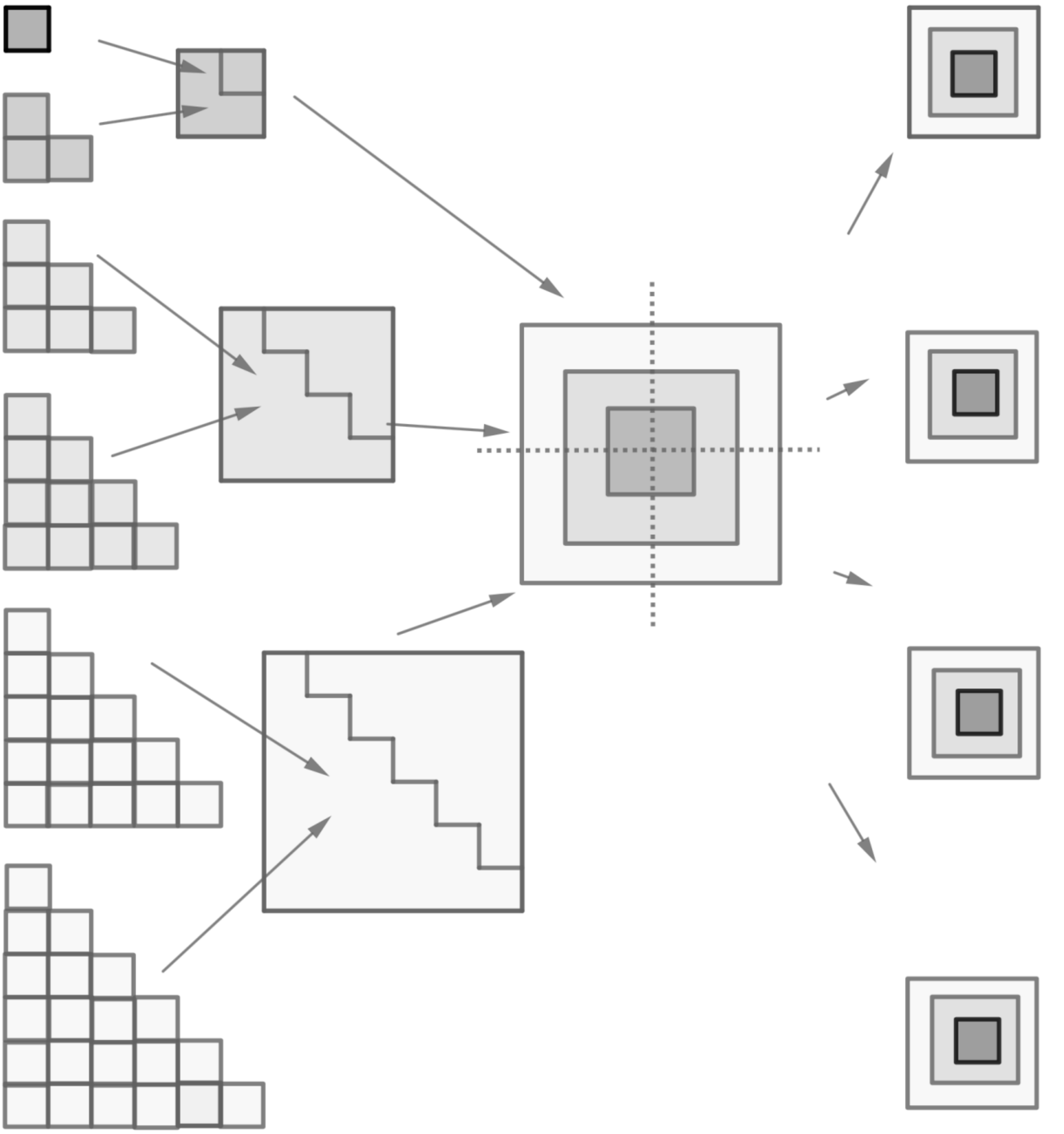


(b) Another identity

$$\begin{aligned}
Tet_{2n} &= (T_1 + T_2) + (T_3 + T_4) + (T_5 + T_6) + \cdots + (T_{2n-1} + T_{2n}) \\
&= \quad S_2 \quad + \quad S_4 \quad + \quad S_6 \quad + \cdots + \quad S_{2n} \\
&= \quad 4S_1 \quad + \quad 4S_2 \quad + \quad 4S_3 \quad + \cdots + \quad 4S_n \\
&= \quad 4(S_1 + S_2 + S_3 + \cdots + T_n) \\
&= \quad 4Pyr_n,
\end{aligned}$$

$\boldsymbol{Tet_{2n} = 4Pyr_n}$. **[8.3]**

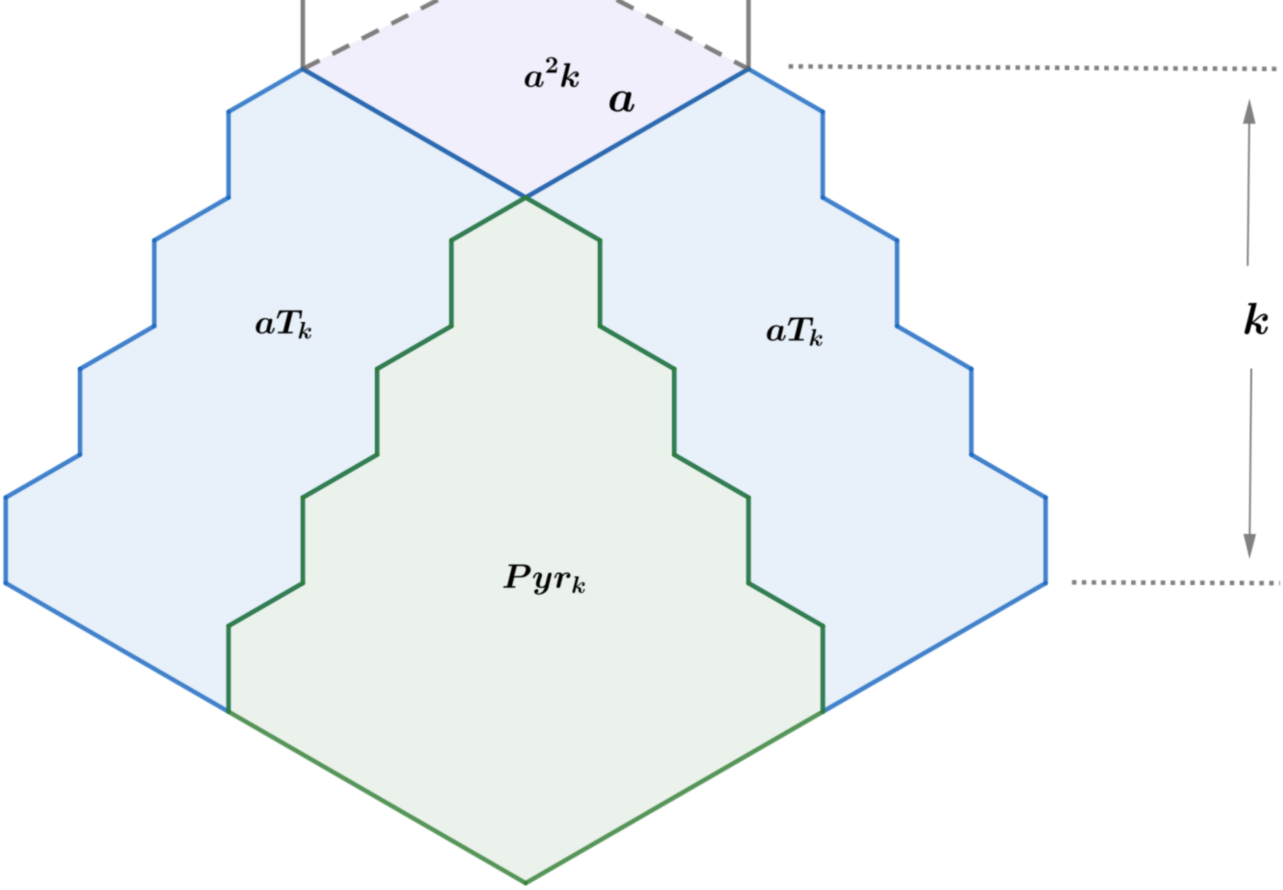
Here is that derivation in visual form:

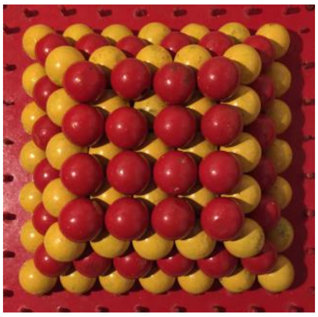


(c) Summing consecutive squares (making the frustum of a pyramid, the general gnomon to a pyramid)

(i) The sum of the consecutive squares $(a+1)^2, (a+2)^2, (a+3)^2, \ldots, (a+k)^2 = Pyr_{a+k} - Pyr_a$, which the algebra shows to be $ak(a+k+1) + Pyr_k$.

That result also emerges from this dissection into 'corner' pyramids:





If we define the cuboidal number $Cu_{m,n}$ as $mn(m+n+1)$, thinking of the factors m, n, $(m+n+1)$ as the dimensions of the cuboid, we can write a figurate equation:

$$\sum_{i=a+1}^{i=a+k} i^2 = \quad \boldsymbol{Pyr_{a+k} - Pyr_a = Cu_{a,k} + Pyr_k} \text{ . } \textbf{[8.4]}$$

Note that $Cu_{m,n}$ is symmetrical in m and n.

Rewriting the equation: $Pyr_{a+k} - Pyr_a - Pyr_k = Cu_{a,k}$,
which we note is symmetrical in a and k, we see that the left side is composite for a or $k > 1$, and that, for a *and* $k > 1$, the expression has ≥ 3 prime factors, not necessarily distinct.

(ii) Two special cases

$a = 2, k = 1$: $Cu_{2,1} = C_2$.

Rearranging the pyramid numbers as sums of squares leads to the unique case where two perfect powers differ by 1: $3^2 = 2^3 + 1$.

$a = 3, k = 2$: $Cu_{3,2} = S_6$.

First rearranging the pyramid numbers as sums of squares, then grouping pairs of consecutive squares as centred square numbers, leads to $CS_5 - C_2 = S_6$.

(iii) Dostor's identity

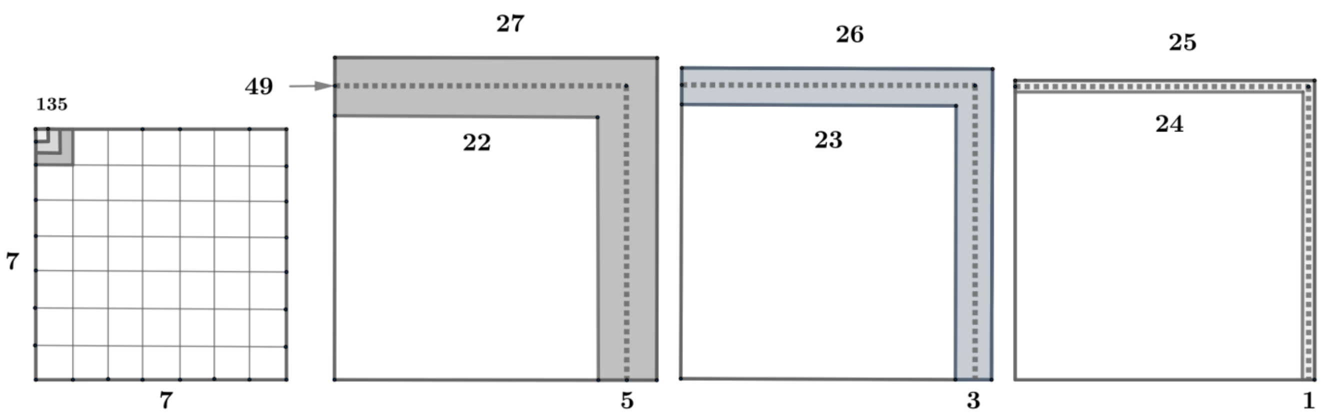
Here is a Pythagorean equation extended as shown by the nineteenth century mathematician Georges Dostor. In figurate terms, we can view the left and right sides of these equations as pyramid frusta.

$3^2 + 4^2 = 5^2$.
$10^2 + 11^2 + 12^2 = 13^2 + 14^2$.
$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$.
... .
$\boldsymbol{(T_{2n})^2 + (T_{2n}+1)^2 + \cdots + (T_{2n}+n)^2 = (T_{2n}+n+1)^2 + (T_{2n}+n+1)^2 + \cdots + (T_{2n+1}-1)^2}$ **[8.5]**

The following visualisation makes the left-most term the subject of the formula. (All it represents in fact is the difference-of-two-squares factorisation of the right hand side.)

$$(T_{2n})^2 = \sum_{i=1}^{i=n} (T_{2n+1} - i)^2 - (T_{2n} + i)^2$$

$$21^2 = (27^2 - 22^2) + (26^2 - 23^2) + (25^2 - 24^2)$$



For an alternative dissection, see Michael Boardman's on p. 92 of Roger B. Nelsen's 'Proofs without Words II'. Boardman makes the middle term the subject.

(iv) The divisibility of sums of consecutive squares

From **[8.4]** we have that the sum of *k* consecutive squares, beginning with the $(a+1)^{th}$ is

$ak(a+k+1)+Pyr_k$.

$Pyr_k = \frac{k(k+1)(2k+1)}{6}$.

(1) If *k* is a prime > 3, it shares no factors with 6, which must therefore divide $(k+1)(2k+1)$, leaving k as a divisor of Pyr_k. Since *k* divides $ak(a+k+1)$ and Pyr_k, it must divide $ak(a+k+1)+Pyr_k$, thus, if *k* is a prime > 3, it must divide the sum of *k* consecutive squares.

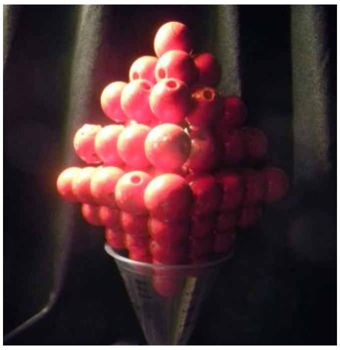
(2) Attending separately to the two parts of the original expression in the way just described, we find that, for $n>1$, 2^{n-1} divides the sum of 2^n consecutive squares.

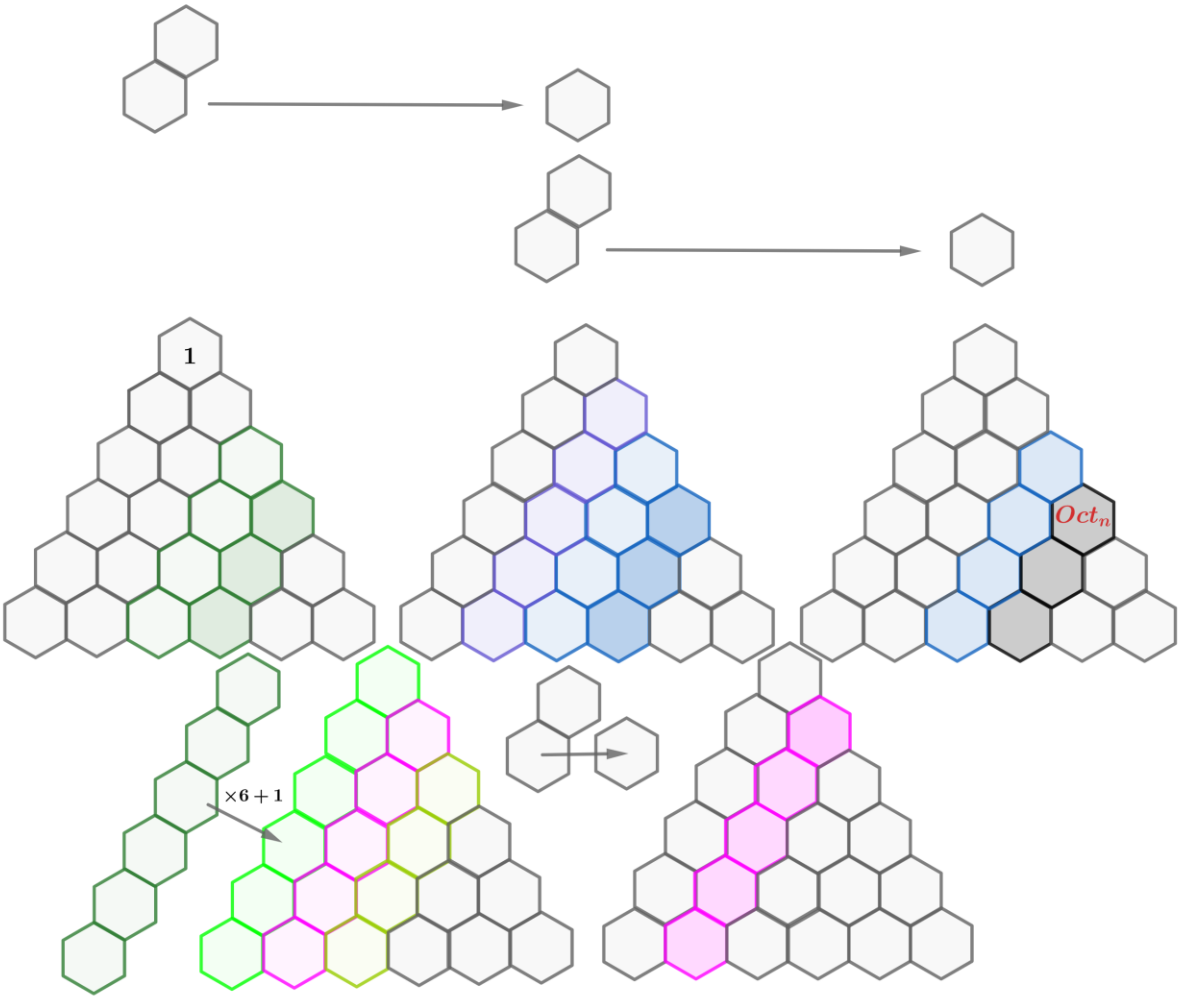
(3) And 3^{n-1} divides the sum of 3^n consecutive squares.

(4) Combining **(1)**, **(2)** and **(3)**, we see that, for *p* a prime > 3 and a > 1, $b>0$, $2^{a-1}3^{b-1}p$ divides the sum of 2^a3^bp consecutive squares.

Chapter 9

The Octahedron, Oct_n





(a) Forming Oct_n

To make an octahedron, we take a pyramid, and stick the base to an inverted pyramid one layer bigger.

$$Pyr_{n-1} + Pyr_n = Oct_n. \quad \textbf{[9.1]}$$

If we take the green cells from the multiplication table we displayed for the pyramid, and pair them side-by-side with those for a pyramid one size smaller, we shall therefore have an octahedron if we sum horizontally, then add those totals. This is shown on the multiplication square below. Note that the diagonal containing the resulting cells is perpendicular to the main diagonal because the dark blue sums are symmetrical about it. (Recall our first multiplication square.)

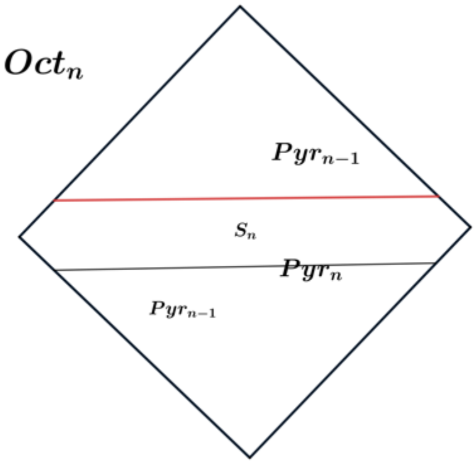
X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
2	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**	**22**	**24**	**26**	**28**	**30**
3	**3**	**6**	**9**	**12**	**15**	**18**	**21**	**24**	**27**	**30**	**33**	**36**	**39**	**42**	**45**
4	**4**	**8**	**12**	**16**	**20**	**24**	**28**	**32**	**36**	**40**	**44**	**48**	**52**	**56**	**60**
5	**5**	**10**	**15**	**20**	**25**	**30**	**35**	**40**	**45**	**50**	**55**	**60**	**65**	**70**	**75**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
7	**7**	**14**	**21**	**28**	**35**	**42**	**49**	**56**	**63**	**70**	**77**	**84**	**91**	**98**	**105**
8	**8**	**16**	**24**	**32**	**40**	**48**	**56**	**64**	**72**	**80**	**88**	**96**	**104**	**112**	**120**
9	**9**	**18**	**27**	**36**	**45**	**54**	**63**	**72**	**81**	**90**	**99**	**108**	**117**	**126**	**135**
10	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**	**110**	**120**	**130**	**140**	**150**
11	**11**	**22**	**33**	**44**	**55**	**66**	**77**	**88**	**99**	**110**	**121**	**132**	**143**	**154**	**165**
12	**12**	**24**	**36**	**48**	**60**	**72**	**84**	**96**	**108**	**120**	**132**	**144**	**156**	**168**	**180**
13	**13**	**26**	**39**	**52**	**65**	**78**	**91**	**104**	**117**	**130**	**143**	**156**	**169**	**182**	**195**
14	**14**	**28**	**42**	**56**	**70**	**84**	**98**	**112**	**126**	**140**	**154**	**168**	**182**	**196**	**210**
15	**15**	**30**	**45**	**60**	**75**	**90**	**105**	**120**	**135**	**150**	**165**	**180**	**195**	**210**	**225**

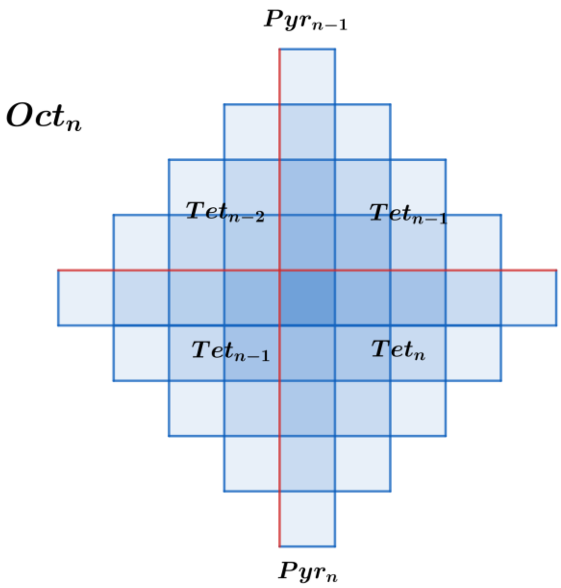
The dark blue cells either side of the central '49' sums to pyramid numbers. We have dissected the octahedron like this:

$$S_n + 2Pyr_{n-1} = Oct_n \, . \quad [9.2]$$

Below left is a schematic vertical section of the model in the photograph above.

Below right is the octahedron represented as a centred square pyramid and seen in plan.





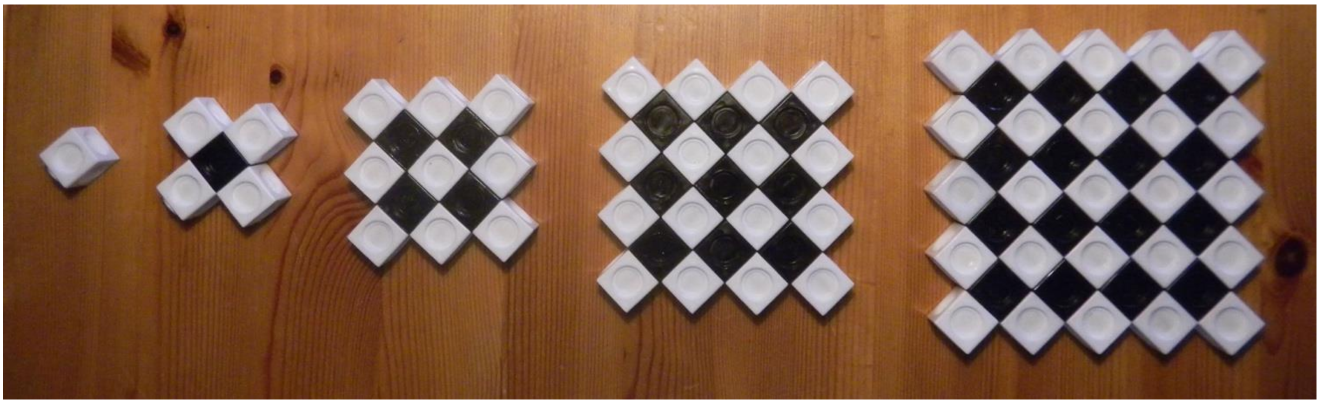
As we know, a centred square is a sum of two consecutive squares. Each layer on the right combines one layer from the upright pyramid with one from the inverted one.

If we isolate the central column, as we did the square in two dimensions, we have the three-dimensional analogue of **[5.2]**:

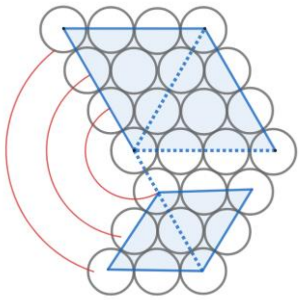
$$\boldsymbol{Oct_n = 4Tet_{n-1} + L_n}. \quad \textbf{[9.3]}$$

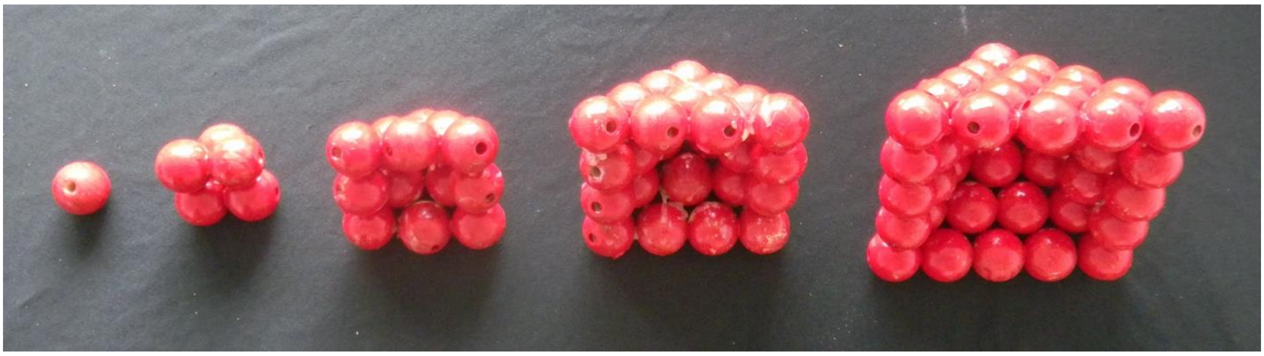
(b) The gnomon

We can also treat the centred square as gnomon to an octahedron. Here is a *CS* sequence:



If we swap the cubes for spheres and push in the centres, we have pyramidal shells which nest to form the octahedron.
Here is a net, showing how
the consecutive squares appear in the spheres
and how the net folds about the blue lines
into a shell.





$\boldsymbol{Oct_{n-1} + CS_n = Oct_n}$**.** **[9.4]**

We give here the grandparent and great-grandparent for information only:

Grandparent:

$$Oct_{n-2} + CS_{n-1} + CS_n$$
$$= \boldsymbol{Oct_{n-2} + 2(2S_n - 4L_n + 3)}$$
$$= \boldsymbol{Oct_n}\textbf{. [9.5]}$$

Great-grandparent, (which can enclose the original):

$$Oct_{n-3} + CS_{n-2} + CS_{n-1} + CS_n$$
$$= \boldsymbol{Oct_{n-3} + 3(2S_n - 6L_n + 19)}$$
$$= \boldsymbol{Oct_n}\textbf{. [9.6]}$$

(c) Packing

Regular tetrahedra pack with regular octahedra to fill space. Likewise tetrahedra and octahedra made from a close packing of spheres. This gives us the identity

$$\boldsymbol{Tet_{2n+1} = 4Tet_n + Oct_{n+1}}. \quad \textbf{[9.7]}$$

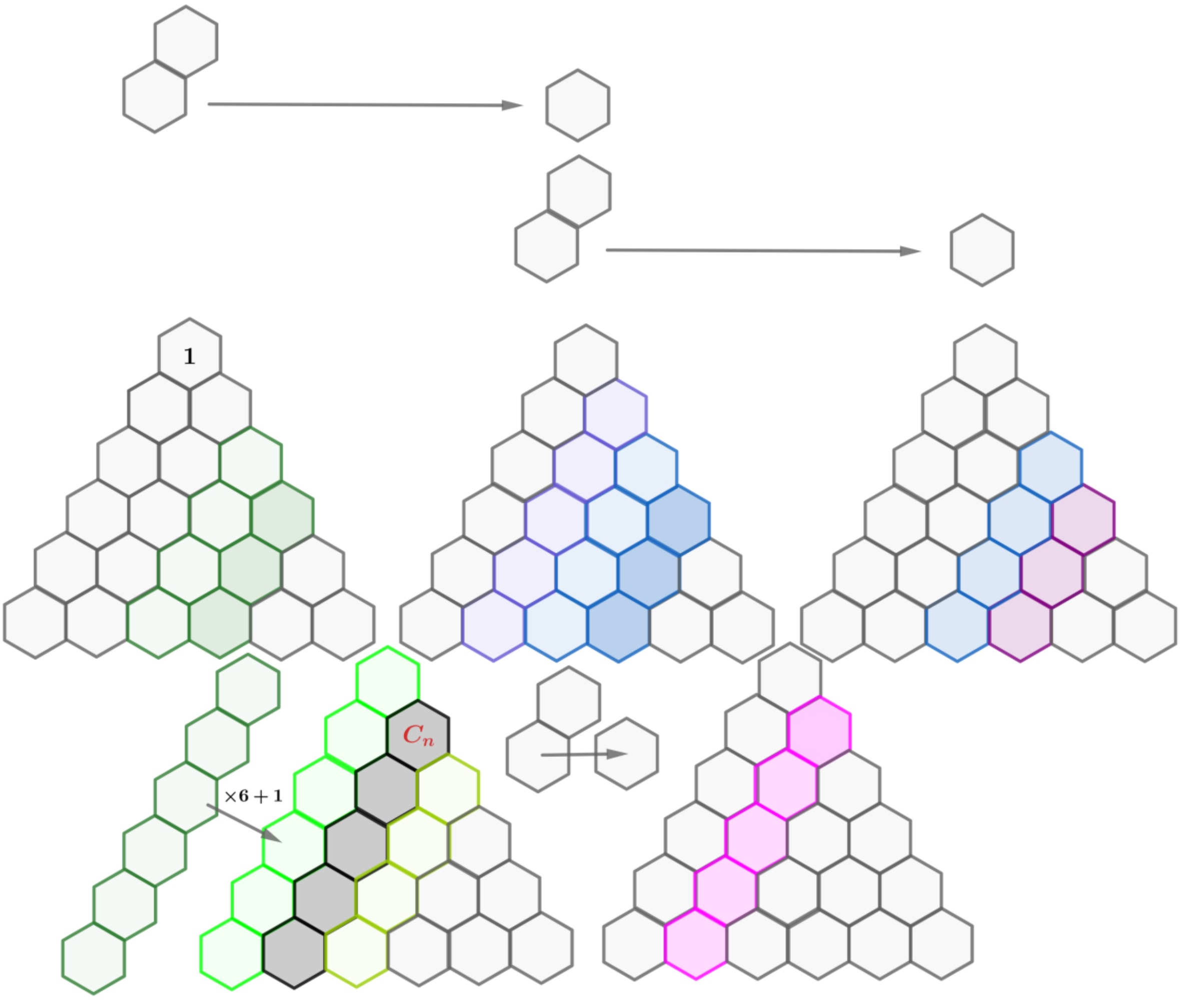
From **(6.2)** we have, with change of suffix, $Oct_{n+1} = 4Tet_n + L_{n+1}$. Substituting in **(9.7)**,

$$\boldsymbol{Tet_{2n+1} = 8Tet_n + L_{n+1}}. \quad \textbf{[9.8]}$$

In the **Cube** chapter, we exploit the fact that we can make a rhombohedron, which is in figurate terms a cube, from two tetrahedra and an octahedron.

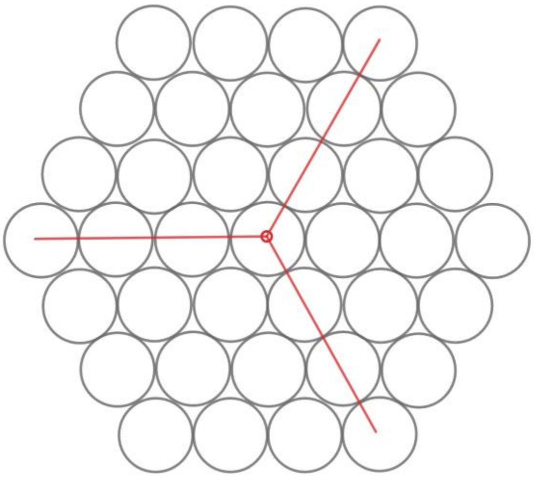
Chapter 10

The Cube, C_n



(a) The gnomon

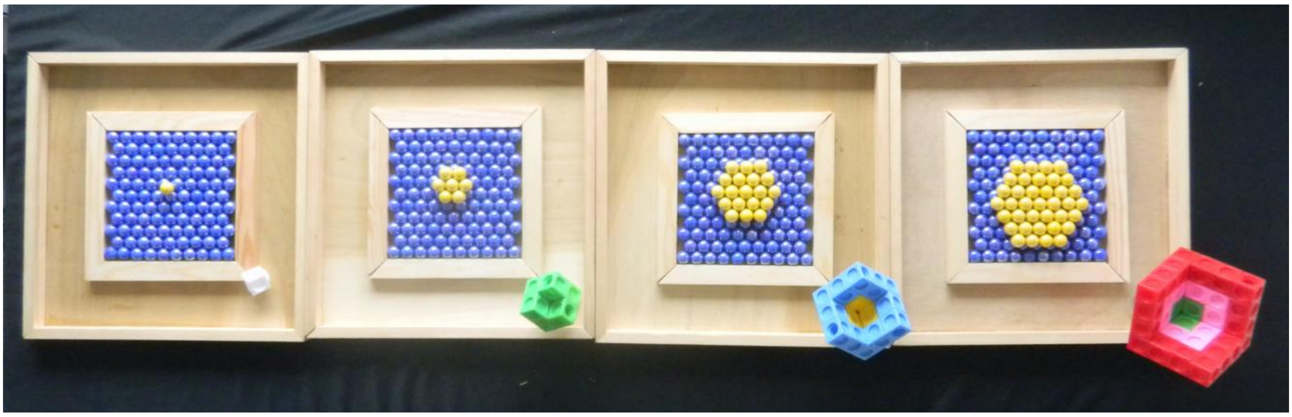
Here again is CH_n.



If we swap the circles for spheres, and fold so that the red lines of spheres at 120° become mutually perpendicular, we shall have the gnomon to a cube. That is to say:

$$\boldsymbol{C_{n-1} + CH_n = C_n}. \textbf{ [10.1]}$$

Here are the gnomons formed from cubes, with colours to bring out the concentric rings in the original centred hexagon, shown in yellow marbles:



Substituting **[6.2]** in **[10.1]**, we have: $\boldsymbol{C_{n-1} + 6T_{n-1} + 1 = C_n}$. **[10.2]**

Building the cube out from a unit in gnomons, we have: $\boldsymbol{6Tet_{n-1} + L_n = C_n}$. **[10.3]**

(b) Other gnomonic relations

The grandparent can be constructed to enclose the cube. The identity is:

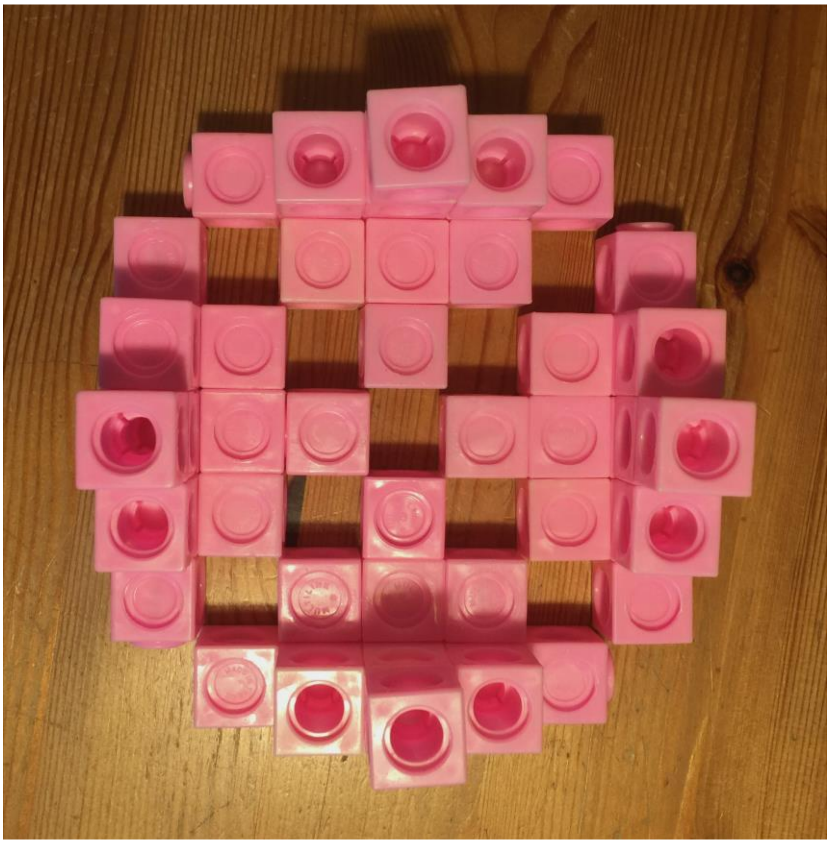
$$\boldsymbol{C_n = C_{n-2} + CH_{n-1} + CH_n = C_{n-2} + 6S_{n-1} + 2}. \textbf{ [10.4]}$$

If the enclosing box is $C_{2n+1} - C_{2n-1}$, we can break it down like this:

$$12CS_n + 24(n-1) + 6 + 8.$$

CS_n for each edge	$(n-1)$ for each half of a face diagonal	1 for each face	1 for each vertex

The picture illustrates this structure:



The great-grandparent identity is:

$$\boldsymbol{C_n} = C_{n-3} + CH_{n-2} + CH_{n-1} + CH_n = \boldsymbol{C_{n-3} + 9(S_n - 3L_n + 3)}. \quad \textbf{[10.5]}$$

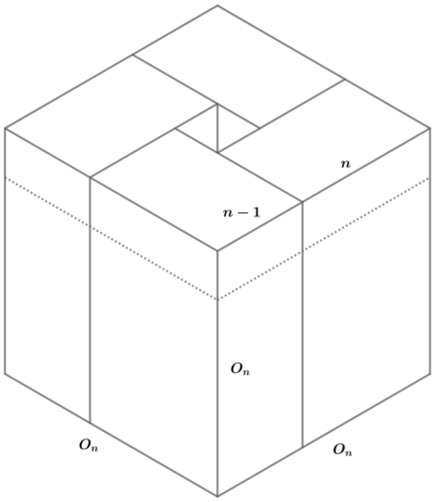
In the **Pyramid** chapter we described a dissection based on that of Man-Keung Siu where 6 pyramids could be arranged as a cuboid. If we arrange four such cuboids like this, you see that, writing $2n-1$ as O_n, we have:

$$\boldsymbol{24Pyr_{n-1} + O_n = C_{O_n}}. \quad \textbf{[10.6]}$$

The dimension of O_n is 3, as it represents a square prism measuring $1 \times 1 \times O_n$.

The part above the dotted line recalls **[3.6]**. We have:

$$\boldsymbol{O_n(8T_{n-1} + 1) = C_{O_n}}. \quad \textbf{[10.7]}$$

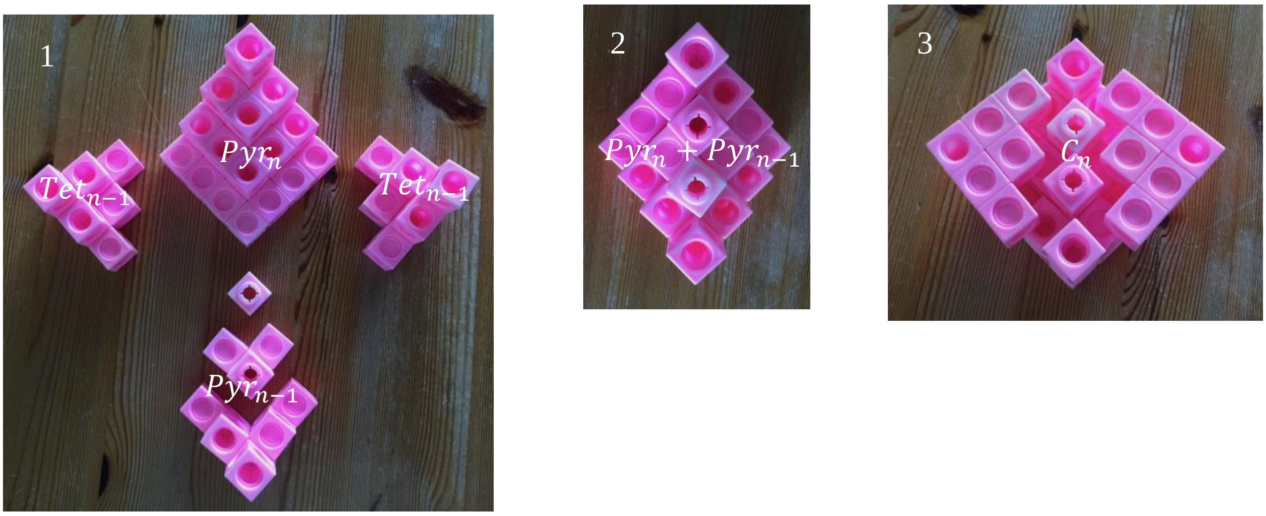


In the **Octahedron** chapter we mention that we can make a rhombohedron, which is in figurate terms a cube, from two tetrahedra and an octahedron. This gives us the following identity:

$$\boldsymbol{2Tet_{n-1} + Oct_n = C_n}. \quad \textbf{[10.8]}$$

Substituting from **[9.1]**: $$\boldsymbol{2Tet_{n-1} + Pyr_{n-1} + Pyr_n = C_n}. \quad \textbf{[10.9]}$$

This is realised in the following dissection, where we replace spheres with cubes, using corner tetrahedra and corner pyramids:

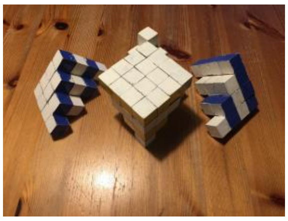
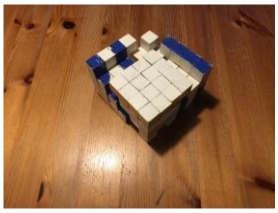
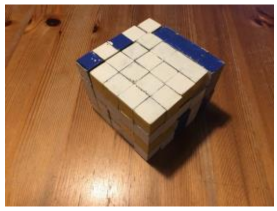


With **[8.2]** we transform **[9.10]** into this: $$\boldsymbol{Tet_{n-2} + 4Tet_{n-1} + Tet_n = C_n}. \quad \textbf{[10.10]}$$
This is the 3-D analogue of **[6.12]**.

Removing one layer from the cube and again using [**8.2]**, we have:

$$\boldsymbol{2(Tet_{n-1} + Pyr_{n-1}) = C_n - S_n}. \quad \textbf{[10.11]}$$

The following dissection, using corner tetrahedra and corner pyramids, shows this:

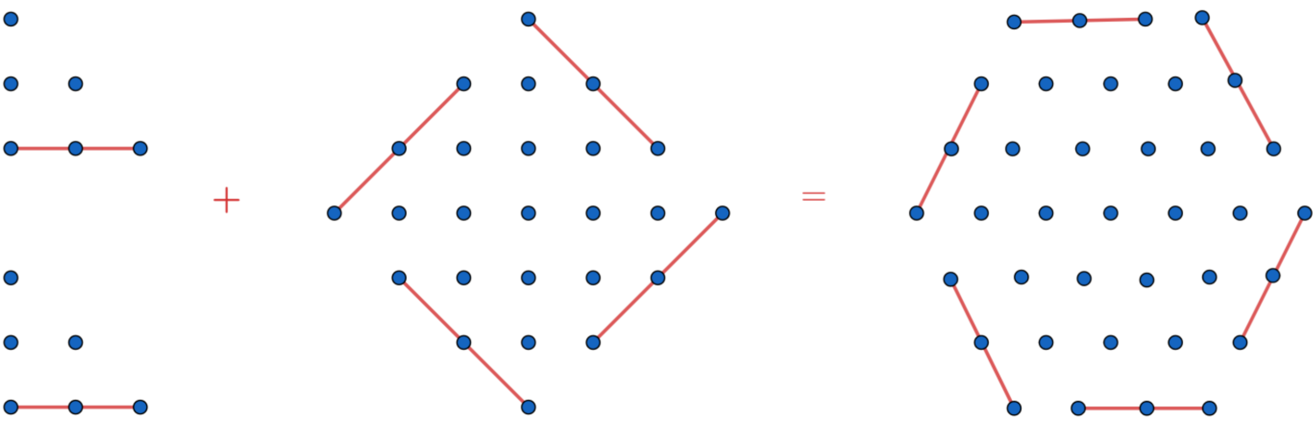
   

We note the analogy between **[6.10]** and **[10.8]**. But it's instructive to track the analogies right back to points:

↓ = 'is gnomon to'

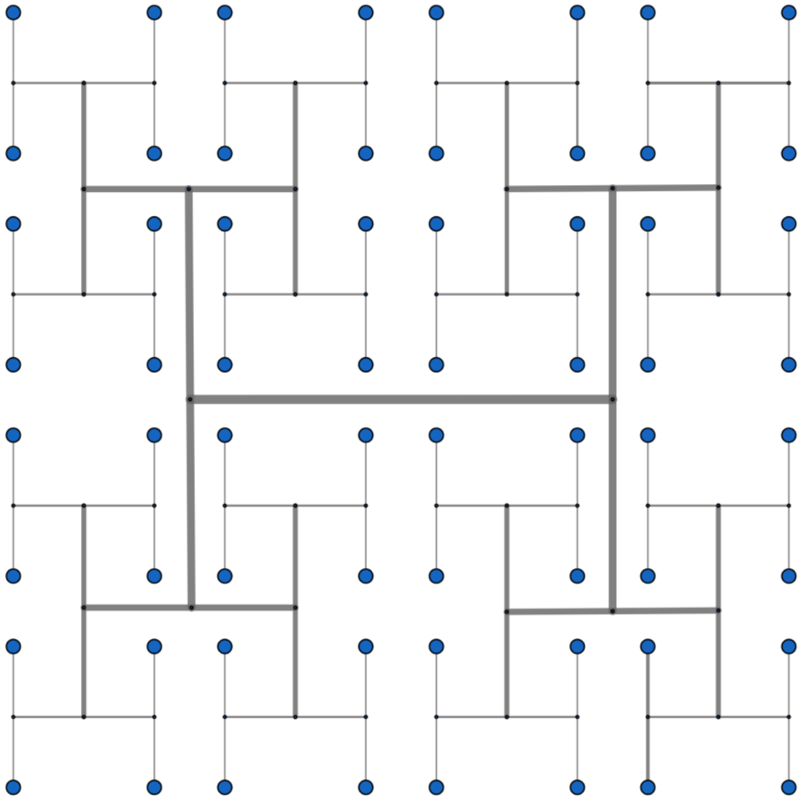
1.	$2(1)$	+	$4(1)$	=	$6(1)$
	↓		↓		↓
2.	$2(L_{n-1})$	+	$4(L_{n-1})$	=	$6(L_{n-1})$
	↓		↓		↓
3.	$2(T_{n-1})$	+	CS_n	=	CH_n
	↓		↓		↓
4.	$2(Tet_{n-1})$	+	Oct_n	=	C_n

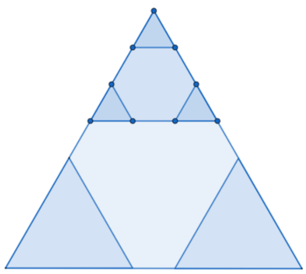
The following figure shows the stage **2** elements being added to complete the stage **3** figures.

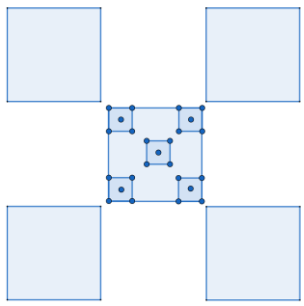


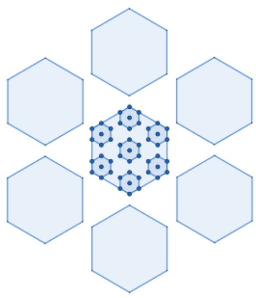
(c) The graphic representation of cubes and higher powers

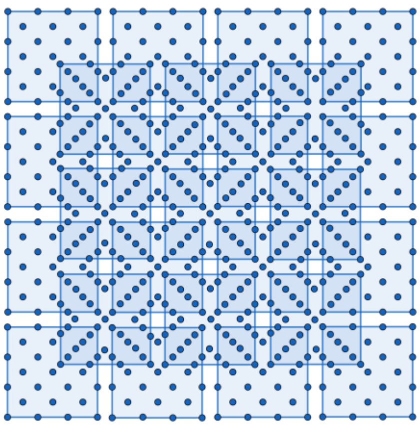
For squares, a pattern of dots in *n* rows and *n* columns is the definitive representation. To represent a power higher than 2 we can find a simple motif and replicate it as a fractal. Below we have shown the cubes of 3, 5 and 7 in this way, and have taken 2 up to the sixth power as an 'H' fractal. 5 is CS_2. 5^2 is CS_4. Making our motif a pair of consecutive squares superposed, we can produce a more compact representation of even powers of 5 than by continuing to replicate the 'quincunx'. Here is 5^4. We also show $(CS_3)^2 = 13^2$ in this way.

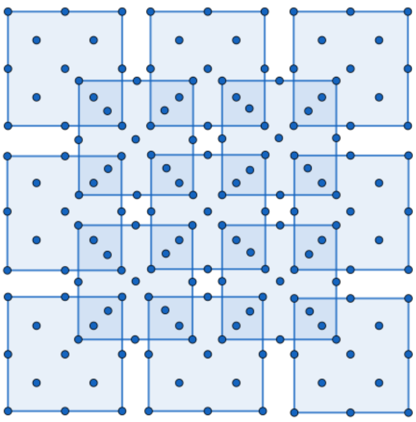






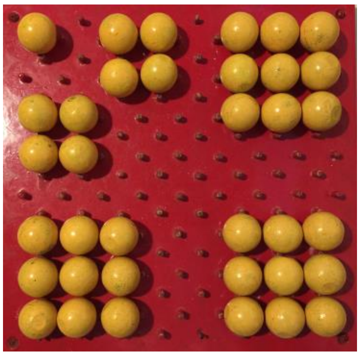


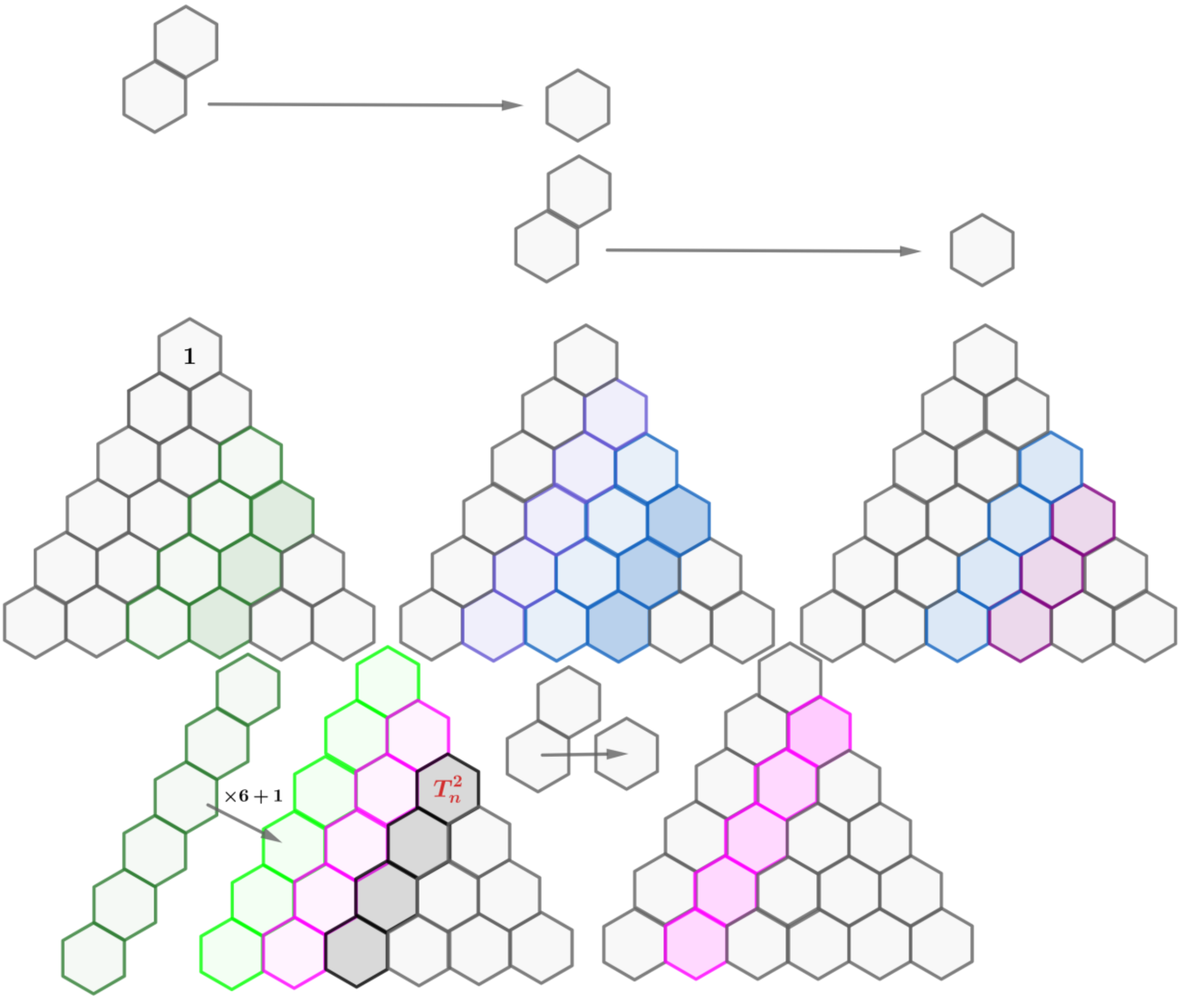




Chapter 11

The square of the triangle, $(T_n)^2$





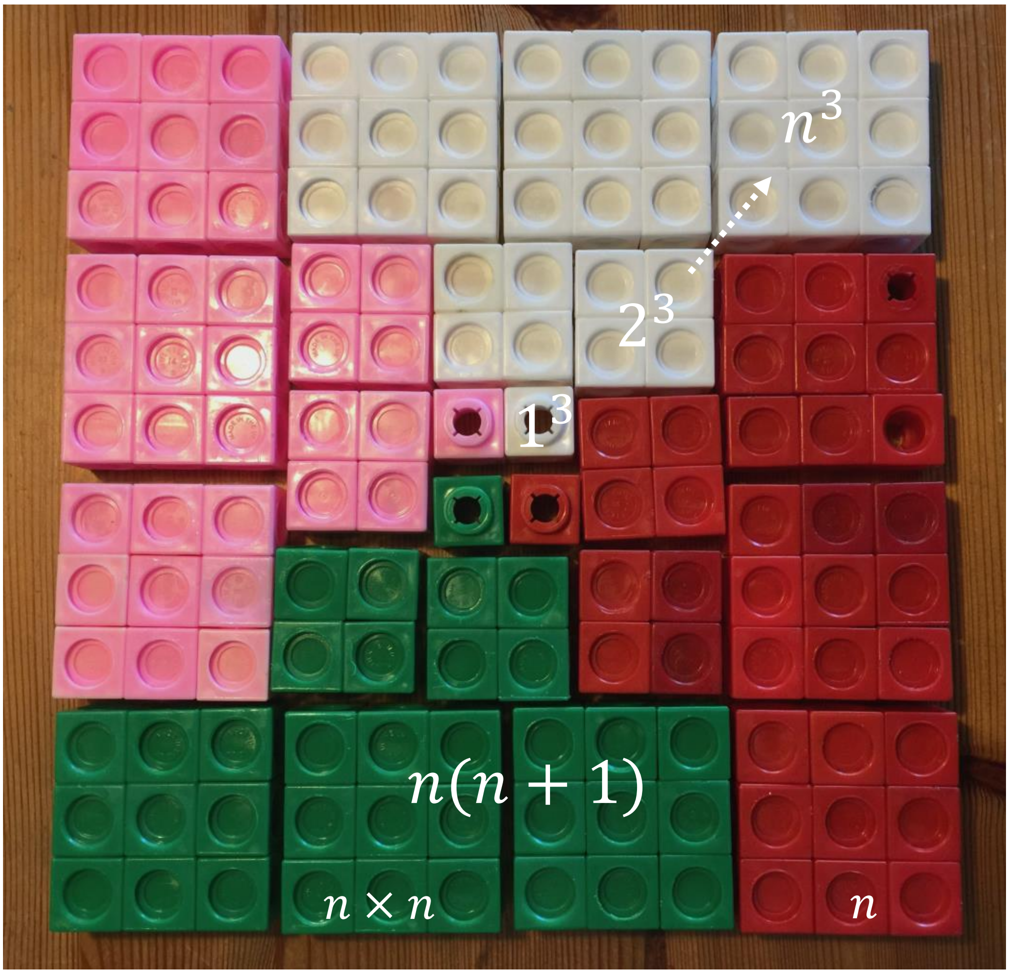
(a) The identity concerned

$$\sum_{i=k+1}^{i=n} \boldsymbol{C_i} = (T_n)^2 - (T_k)^2 = (\boldsymbol{T_n} + \boldsymbol{T_k})(\boldsymbol{T_n} - \boldsymbol{T_k}),\ n \geq k+1. \quad \textbf{[11.1]}$$

There are many graphical representations of this result for the case $k = 0$, when we have simply

$\sum_{i=1}^{i=n} i^3 = (T_n)^2$.

See especially Roger B. Nelsen's 'Proof without Words', p. 87, where there is a dissection found independently by Antonella Cupillari and Warren Lushbaugh:



Since $n \geq k + 1$, **[11.1]** shows that any sum of consecutive cubes must be a composite number.

(b) Special cases

(i) The case $k = 2, n = 5$ records a run of cubes, the fourth of which is the sum of the previous three. (There is a celebrated dissection puzzle showing this result.)

$C_3 + C_4 + C_5 = C_6 = (T_5)^2 - (T_2)^2 = (T_5 + T_2)(T_5 - T_2) = (T_3)^3$.

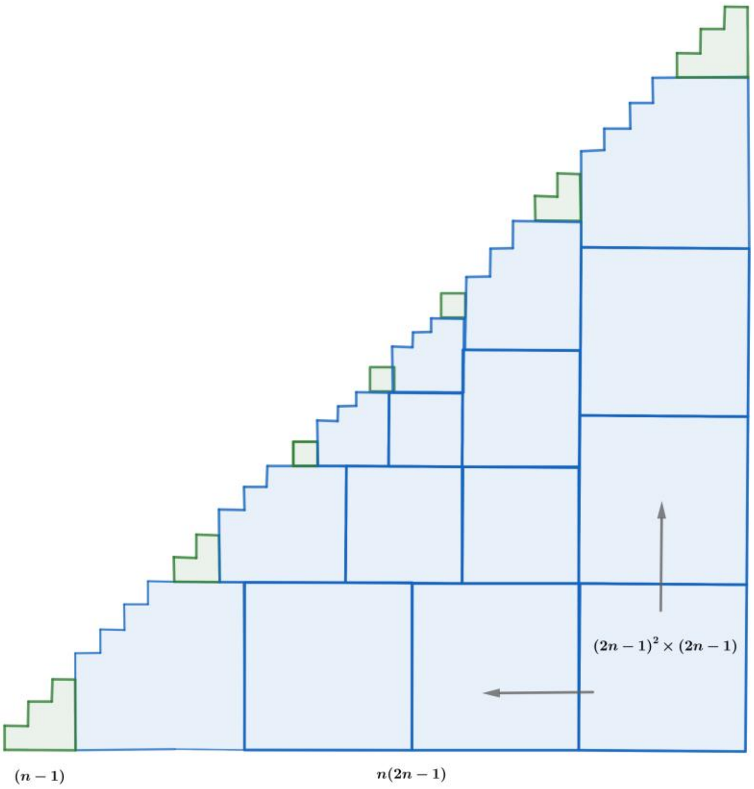
Setting up this equation: $(n-2)^3 + (n-1)^3 + n^3 = (n+1)^3$, we derive:
$(n-5)(n^2 - n + 1) = 0$, whose only integer root is 5. The run of cubes is therefore unique.

(ii) Taking $k = 0$, we can ask whether the sum of the first *n* cubes can be a cube.

We know that it is the square of a triangle number, so we have:
$(T_n)^2 = m^3$,
$T_n = m^{\frac{3}{2}}$.
For this to be an integer, we require $m = s^2$, whence
$T_n = s^3$.
But we saw in chapter **1**, section **(i)**, that no triangle number > 1 is a cube.
Therefore the sum of the first *n* cubes cannot be a cube.

(iii)

If we add the first *n* consecutive *odd* cubes, we obtain T_{2n^2-1}. $\sum_{i=1}^{i=n}(\mathbf{2i-1})^{\mathbf{3}} = \boldsymbol{T_{2n^2-1}}$. $[\mathbf{11.2}]$
On p. 91 of Roger B. Nelsen's 'Proofs without Words' is a demonstration of this fact due to Monte J. Zerger:



By analogy, the sum of the first *n* consecutive odd *squares* is:

1 + 9 + 25 + ... + $S_{2n-1} = 1 + (3 + 6) + (10 + 15) + \cdots + T_{2n-2} + T_{2n-1}$
$= 1 + 3 + 6 + \cdots + T_{2n-1} = Tet_{2n-1}$, $\sum_{i=1}^{i=n}(\mathbf{2i-1})^{\mathbf{2}} = \boldsymbol{Tet_{2n-1}}$. **[11.3]**

We saw in section **(b), (ii)** that the first *n* cubes cannot sum to a cube. From **[8.2]** we learn that the sum of the first *n odd* cubes is a triangle number. Since a triangle number > 1 cannot be a cube, it is also true that the sum of the first *n odd* cubes cannot be a cube.

The sum of the first *n even* cubes = $2^3 \times$ the sum of the first *n* cubes $= 2^3(T_n)^2$. Since T_n cannot be a cube > 1, the right side of this equation cannot be a cube. So it is also true that the sum of the first *n even* cubes cannot be a cube.

(iv) [8.2] gives us the sum of the consecutive odd cubes $(b+1)^3$ +$(b+2)^3 + (b+3)^3 + \cdots + a^3$:

$T_{2a^2-1} - T_{2b^2-1}$, which simplifies to $(a+b)(a-b)[2(a^2+b^2)-1]$.

The third bracket is odd; the first two brackets have the same parity. When a, b are both even, the expression has the factor 4, that is to say, an even number of consecutive odd cubes divides by 4.

When $a = b+1$, we have C_{2b+1}.

(c) The divisibility of sums of consecutive cubes

Consider sequences of consecutive cubes which do not necessarily start with 1. We have:

$\sum_{i=a+1}^{i=a+k} i^3 = (T_{a+k})^2 - (T_a)^2$, for $a \geq 0$, which simplifies to $[ka + T_k][a^2 + (k+1)a + T_k]$.

If we are interested in finding divisors of those sums, there are several useful principles we can apply.

(1) Extracting common factors, we have immediate divisors of those sums. But, by regrouping terms, we can test whether 2 and 3 are further divisors. This move rests on the distributivity of multiplication over addition. If p divides a, say $a = fp$, and p divides b, say $b = gp$, then p divides $(a+b)$ since $(a+b) = p(f+g)$**.** So, if we have a bracket $(x+3t)$, we have only to show that 3 divides x to know that 3 divides the whole bracket, $(x+3t)$. For example, when *k* = 3, we have: $3(a+2)(a^2+4a+6)$. We rewrite the expression, recasting the second bracket: $3(a+2)[a(a+1)+3(a+2)]$. Exactly one of $a, (a+1), (a+2)$ must divide by 3. Therefore either $(a+2)$ or $a(a+1)$ must have factor 3. Therefore 3 divides either $(a+2)$ or $[a(a+1)+3(a+2)]$. So the original expression must have divisor 3^2.

(2) It is often useful to isolate $a(a+1)$ for we know that the two factors have opposite parity and therefore that $a(a+1)$ is even.

(3) We can invoke another consequence of the distributivity of multiplication over addition.
If a sequence of length *k* has divisor *d*, then a sequence of length *nk* must also have divisor *d*.
This is an instance of the following more general result.

Let $d(a)$ be a known divisor of *a*. Let $d(b)$ be a known divisor of *b*. Then we know that $lcm(a,b)$ has a divisor ≥ $lcm[d(a), d(b)]$.

(4) Odd and even terms alternate. This means that the sum of 4 consecutive cubes will be even since we shall be adding two even terms and two odd. But the sum of six consecutive cubes will be odd since we shall be adding three even terms and three odds. Generalising, the sum of $k = 4n+2$ cubes will be odd for *n* > 0. The sum of an odd number of consecutive cubes will be odd or even depending on the parity of the first cube:

Total even

Odd Even Odd Even Odd Even Odd ...

Total odd

There are a number of observations of interest.

(i) When *k* is a odd, the first bracket in our general expression = $\left[ka + \frac{k(k+1)}{2}\right]$. Since (k+1) is even, the second term is an integer and we can take out the factor k. So, when *k* is odd, *k* divides the sum of *k* consecutive cubes.

(ii) When $k = 2^n$, we have this total:
$[2^n a + T_{2^n}][a^2 + (2^n + 1)a + T_{2^n}]$, which simplifies to:
$2^n[2a + 2^n + 1][T_a + 2^{n-2}(2a + 2^n + 1)$, that is, 2^n divides the sum of a sequence of 2^n consecutive cubes where *n* > 1.

(iii) By principle **(3)** we can combine results **(i)** and **(ii)** and say that $2^n(2s+1)$ divides the sum of $2^n(2s+1)$ consecutive cubes where *n* > 1.

(iv) When $k = 3^n$, we have
$[3^n a + T_{3^n}][a^2 + (3^n + 1)a + T_{3^n}]$
$= 3^n\left[a + \frac{3^n+1}{2}\right]\left[a(a+1) + 3^n\frac{3^n+1}{2}\right]$
$= 3^n\left[(a+2) + \frac{3^n-3}{2}\right]\left[a(a+1) + 3^n\frac{3^n+1}{2}\right]$
$= 3^n\left[(a+2) + 3\frac{3^{n-1}-1}{2}\right]\left[a(a+1) + 3^n\frac{3^n+1}{2}\right]$.

Having isolated those two multiples of 3 in the square brackets, we invoke the argument we used above that either $(a+2)$ or $a(a+1)$ has divisor 3 and therefore that either the first or second square bracket has factor 3, concluding that the whole expression has factor 3^{n+1}. So, when $k = 3^n$, $n > 0$, 3^{n+1} divides the sum of 3^n consecutive cubes.

(v) We can combine **(iii)** and **(iv)** to make a further generalisation. When $k = 2^a 3^b(2s+1)$, $2^a 3^{b+1}(2s+1)$divides the sum of the $2^a 3^b(2s+1)$ consecutive cubes where *a* > 1, $b > 1$. For example, 36 divides the sum of 12, 45 the sum of 15, 54 the sum of 18, 63 the sum of 21, 72 the sum of 24 consecutive cubes.

Principle **(3)** allows us to check particular results we derive.

(d) When is the sum of consecutive cubes a square?

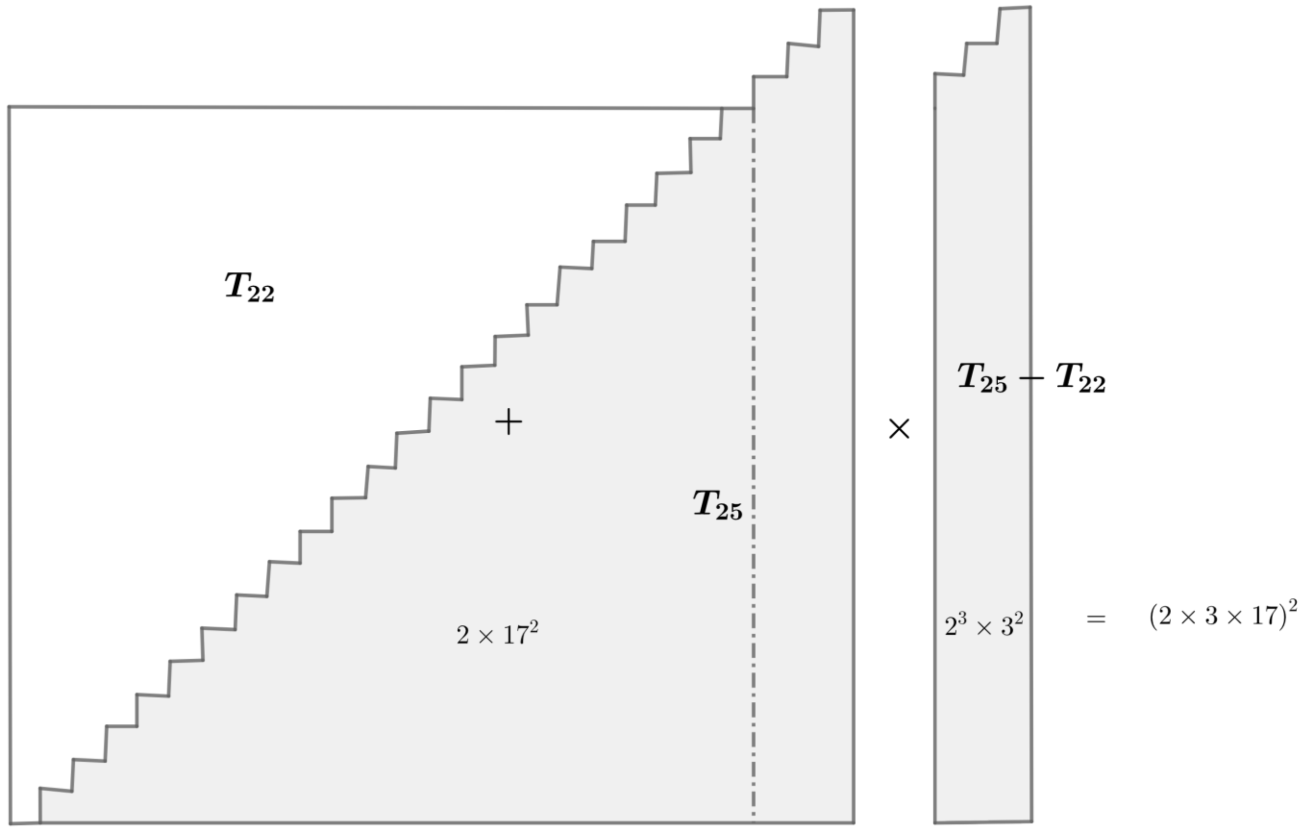
Section **(c)** opened with this equation for the sum of *k* consecutive cubes, beginning with the $(a+1)^{th}$:
$[ka + T_k][a^2 + (k+1)a + T_k]$.
Regrouping terms, we have:
$[k(a+1) + T_{k-1}][(a+k)(a+1) + T_{k-1}]$.
Comparing corresponding terms in the two square brackets, we see that the sum is certainly a square if $a + k = k$, i.e. $a = 0$.

But are there other cases?

If $a > 0$, we can write, for some rational number $s > 1$,
$[(a+k)(a+1) + T_{k-1}] = s^2[k(a+1) + T_{k-1}]$.
If $k = 1$, we have $s^2 = a + 1$. We can then find squares by choosing *a* values 1 less than a square. For example, when $a = 3$, we have $4^3 = 8^2$; when $a = 8$, $9^3 = 27^2$ and so on.

To test for $k > 1$, we return to **[11.1]**. T_{a+k} and T_a are respectively the hypotenuse and a shorter side of a Pythagorean triangle. The square we require is that of the remaining side, *d*.

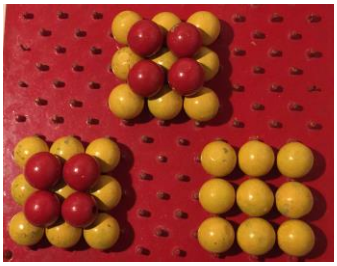
Here is the smallest solution, showing $d^2 = (T_{a+k} + T_a)(T_{a+k} - T_a)$. The task of finding this was set by the famous puzzlist Charles Dudeney.

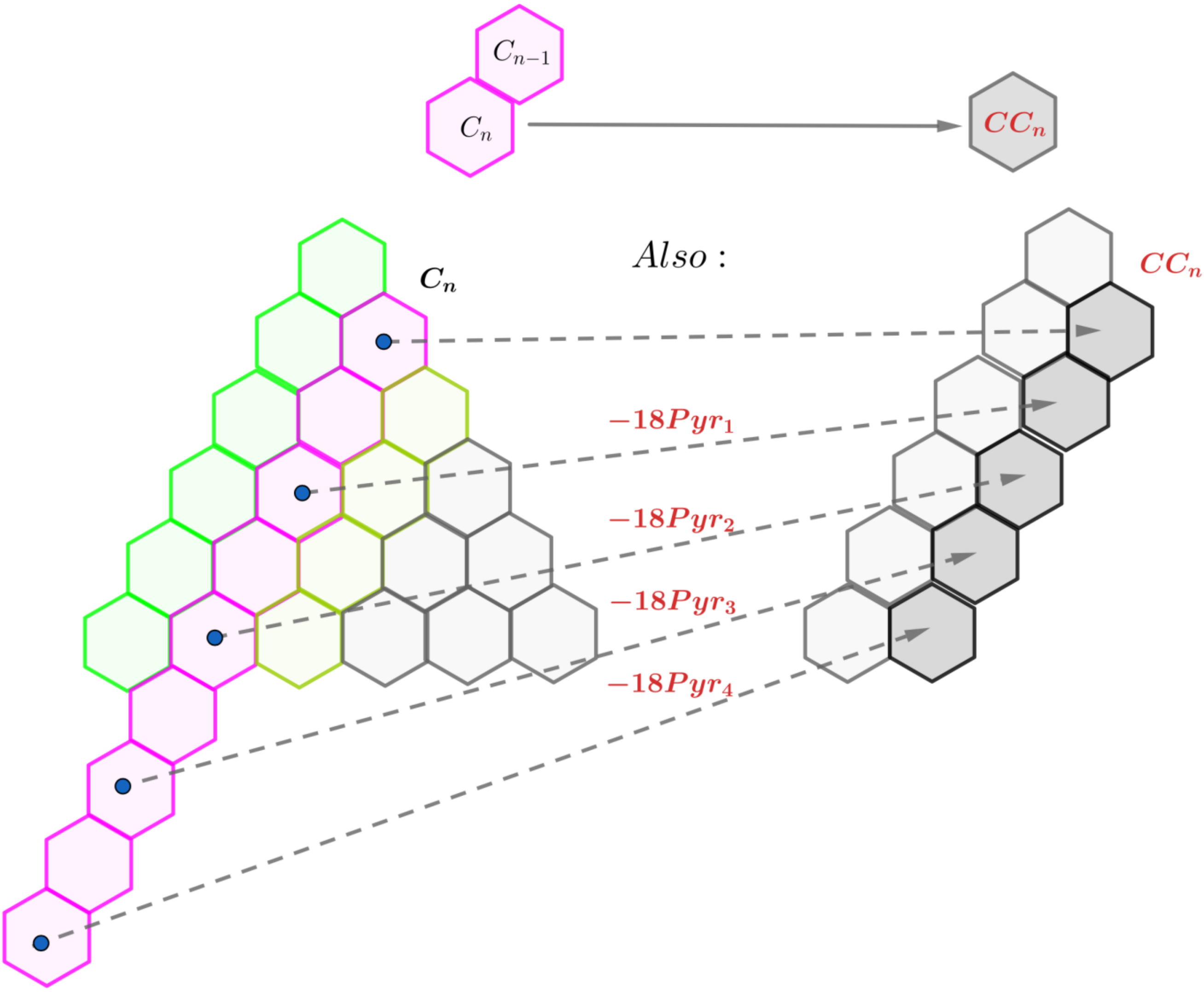


$a = 22, k = 3, d = 204.$ The triple is (204, 253, 325). (Out of interest, this result correspond to an *s* value in our previous formula of $\frac{17}{6}$.)

Chapter 12

The Centred Cube, CC_n



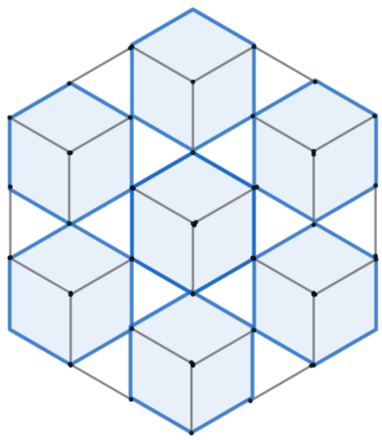


For two consecutive cubes, we have the n^{th} centred cube, CC_n, $=$
$(n-1)^3 + n^3 = (T_n)^2 - (T_{n-2})^2 = 2n^3 - 3n^2 + 3n - 1 = (2n-1)(n^2 - n + 1)$.

Note that the number is necessarily composite for $n > 1$, a result generalised in our note to **[11.1]**.

Reaching back to **[3.7]** to translate the two brackets, we have: $\boldsymbol{CC_n = O_n(T_{n-2} + T_n)}$ **[12.1]**

We can build it out from a unit cube as a crystal lattice of the body-centred cubic type. The cubes share only vertices. (In an analogous way, so do the squares in the checkerboard representation of CS_n.)

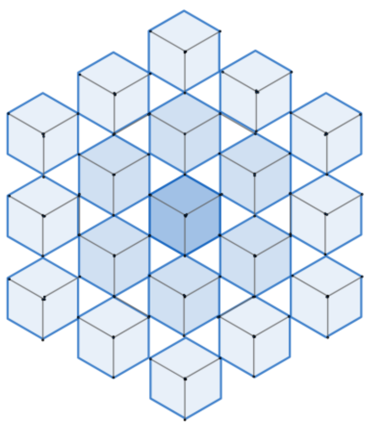


The figure shows CC_2. We look down a 3-axis. Thus the central cube and the one behind it are hidden. There is a cube for each vertex of the central cube, therefore $1 + 8 = 9$ in total.

We can also think of the figure as a 3-cube from which we've subtracted a cube for each face and each edge of the central cube, again giving $3^3 - (6 + 12) = 9$.

We shall develop these two approaches, generating CC_n first by addition, second by subtraction.

Here is CC_3 from the same viewpoint:



Thinking in terms of CH numbers, the outer ring just account for 1 cube; to the inner ring, we must add 2; to the central ring, a further 2, giving a total of:

$2CH_1 + 2CH_2 + CH_3 = 35$.

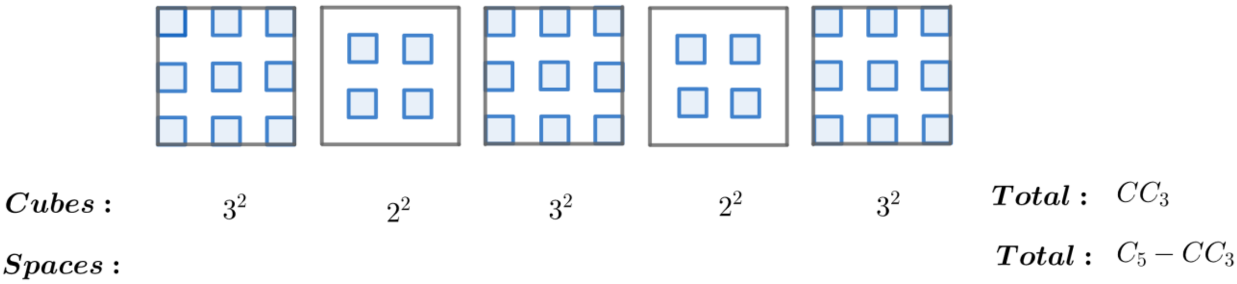
This generalises as:

$$\boldsymbol{CC_n = 2\sum_{i=1}^{i=n-1} CH_i + CH_n}. \quad \textbf{[12.2]}$$

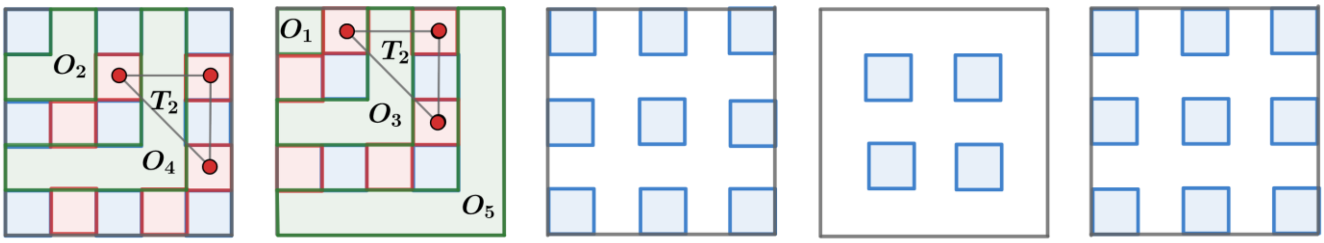
Bearing in mind **[10.1]**, we see how this expression yields a sum of two consecutive cubes.

To perform a subtraction, we start with our outer cube, C_{2n-1}, and take away what the algebra shows to be Pyr_{n-1} lots of $(6 + 12)$.

We illustrate this result for $n = 3$, taking sections perpendicular to a 4-axis.



Having found the number of spaces in one way, we now do so in another:



We need four identities. **[2.2]**, combined with the definition of a triangle number, allows us to write the sum in terms of triangle numbers. **[3.8]** and **[3.9]** enable us to write these in terms of T_1 and T_2. By **[3.5]** we translate the total into $18(S_1 + S_2)$, which is $18Pyr_2$. Convince yourself that the argument generalises.

But we demonstrate the result more simply below.

Our formula is thus: $\boldsymbol{CC_n = C_{2n-1} - 18Pyr_{n-1}}$. **[12.3]**

This is analogous to **[5.3]**:

$$CS_n = S_{2n-1} - 4T_{n-1}.$$

In place of the 12 edges and 6 faces of the cube, we have the 4 sides of the square.

Adding the expressions for two consecutive cubes from **[9.9]**, we have:

$$\boldsymbol{2Pyr_{n-1} + Oct_{n-1} + Oct_n = CC_n}. \quad \textbf{[12.4]}$$

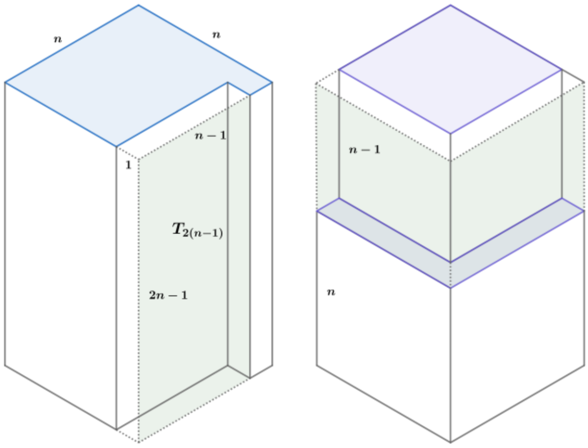
Concerning the difference of two centred cubes, the algebra throws up one interesting identity:

$$\boldsymbol{CC_{n+3} - CC_n = 18(S_{n+1} + 1)}. \qquad \textbf{[12.5]}$$

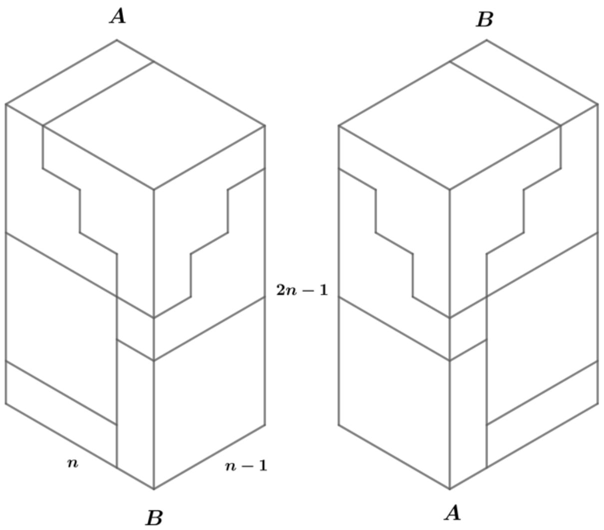
Confirm that the general formula is: $CC_a - CC_b = (a-b)[2(a^2 + ab + b^2) - 3(a + b - 1)]$.
[9.4] corresponds to the case $a = b + 3$.

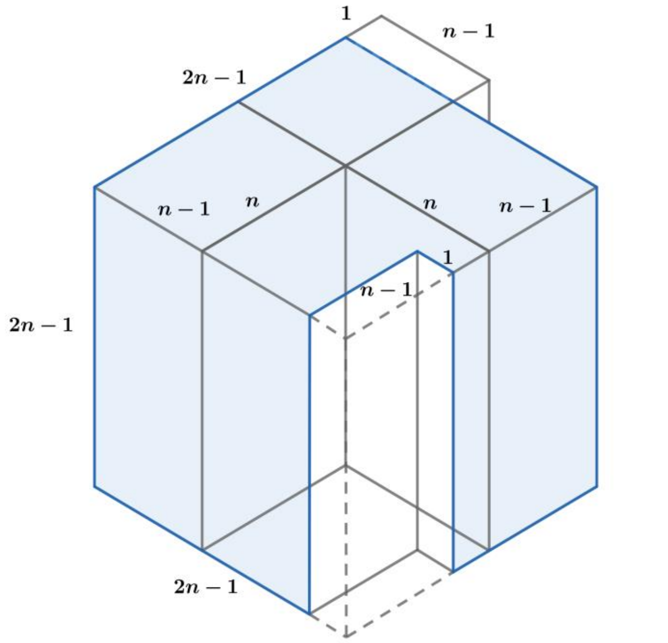
A further identity is: $\boldsymbol{CC_n = O_nS_n - T_{2(n-1)}}$. **[12.6]**

This describes a square prism from which a rectangular prism has been cut. The consecutive defining cubes are shown on the right for comparison. Note that the green part on the left is of equal volume to that on the right.



We use this equivalence to demonstrate **[12.3]**. We begin with the dissection of Man-Keung Siu, which leads to this cuboid, containing 6 pyramids. Arranging three cuboids and the prism as shown on the right, we find that the assemblage exceeds the required cube by exactly the amount by which it is deficient.





The centred cube to modulo 3 & modulo 4

See the table beneath. Since CH_n is gnomon to a cube, and $CC_n = C_{n-1} + C_n$, we can construct the cycle for C_n by deriving a cumulative total from the first line, and CC_n by adding consecutive pairs of values in the second line.

To modulo 3:

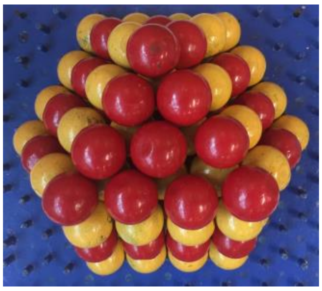
CH_n:	1	1	1	1	1	1	1	1	1	...
C_n:	1	2	0	1	2	0	1	2	0	...
CC_n:	1	0	2	1	0	2	1	0	2	...

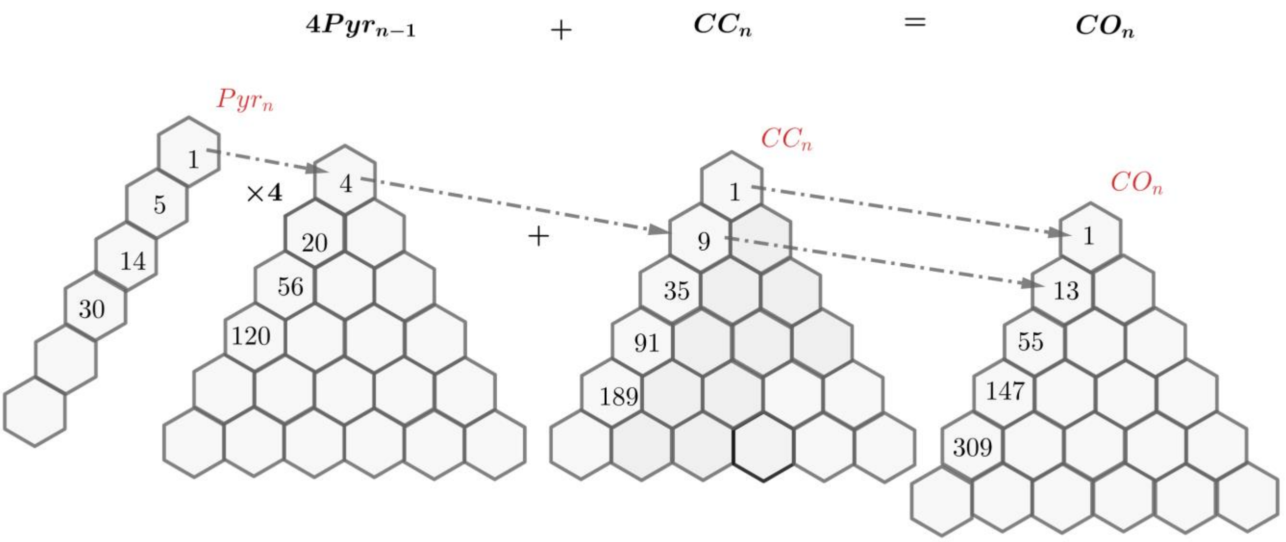
To modulo 4

CH_n:	1	3	3	1	1	3	3	1	1	3	3	1	...
C_n:	1	0	3	0	1	0	3	0	1	0	3	0	...
CC_n:	1	1	3	3	1	1	3	3	1	1	3	3	...

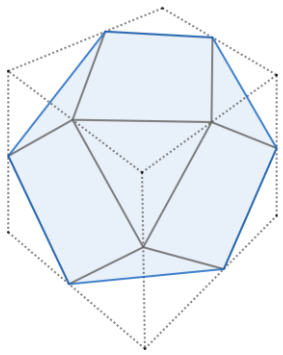
Chapter 13

The Cuboctahedron, CO_n

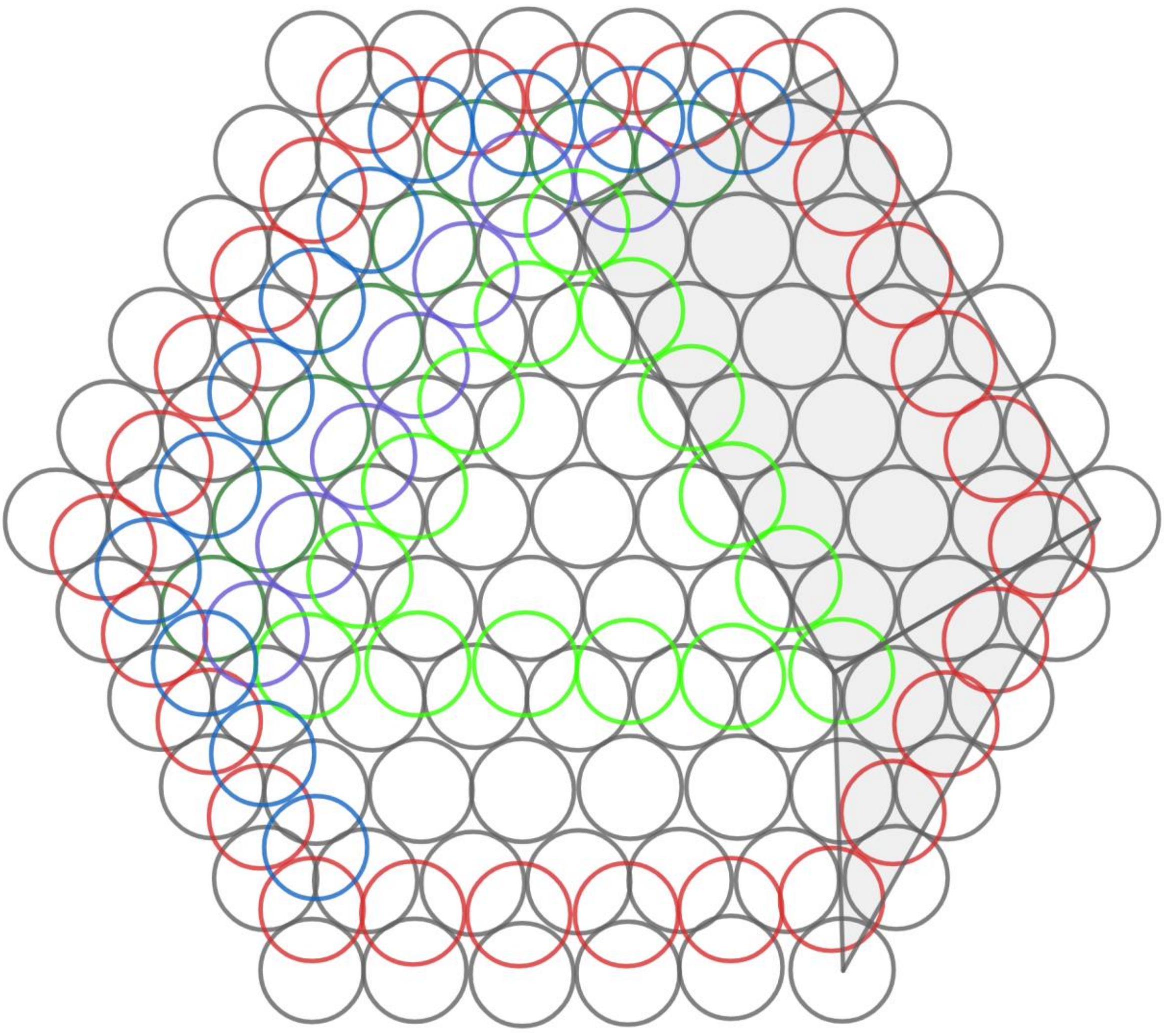




The regular octahedron and cube are dual solids.
We shall take the hybrid form, the cuboctahedron, the solid of interpenetration of the two.
It is a uniform solid with 8 triangular and 6 square faces. We shall carve it out of a close packing of spheres.



In the following view we look down on a triangular face and colour-code horizontal layers.



Taking the numbers of spheres in each row of each layer in cuboctahedra of consecutive size, we have:

1

1 2
2 3 2
1 2

1 2 3
2 3 4 3
3 4 5 4 3
2 3 4 3
1 2 3

1 2 3 4
2 3 4 5 4
3 4 5 6 5 4
4 5 6 7 6 5 4
3 4 5 6 5 4
2 3 4 5 4
1 2 3 4

1 2 3 4 5
2 3 4 5 6 5
3 4 5 6 7 6 5
4 5 6 7 8 7 6 5
5 6 7 8 9 8 7 6 5
4 5 6 7 8 7 6 5
3 4 5 6 7 6 5
2 3 4 5 6 5
1 2 3 4 5

In the five sets, we've picked out in red numbers summing to a CH number. In the last example we've shown that these palindromic runs also occur in a gnomonic arrangement.

To see how the CH numbers arise, label as follows:

5	6	7	8	9	8	7	6	5
$n+1$				$2n+1$	$2n$			$n+1$
	$T_{2n+1}-T_n$					$T_{2n}-T_n$		

$(T_{2n}+T_{2n+1})-2T_n$

From **[4.4]** we can write this as $S_{2n+1}-2T_n$.
Substituting from **[3.5]**, $S_{2n+1}=8T_n+1$, we have the result that the palindromic run = $8T_n+1-2T_n=6T_n+1=CH_{n+1}$ from **[6.2]**.

Because these numbers are gnomons to a cube, we can label the top left array below n^3 and the bottom left array $(n-1)^3$.

The top right and bottom right arrays are equal. We shall show that each $=2Pyr_{n-1}$. We choose labels as below:

1 2 3 4 5 |
2 3 4 5 6 | 5
3 4 5 6 7 | 6 5
4 5 6 7 8 | 7 6 5
5 6 7 8 9 | 8 7 6 5
4 5 6 7 | 8 7 6 5
3 4 5 6 | 7 6 5
2 3 4 5 | 6 5
1 2 3 4 | 5

5	6	7	8	
$n+1$			$2n$	$T_{2n}-T_n$
5	6	7		$T_{2n-1}-T_n$
5	6			.
				.
5				$T_{n+1}-T_n$
Sum:				$Tet_{2n}-Tet_n-nT_n$

We confirm from the algebra that this expression$=2Pyr_{n-1}$.

This gives us a grand total for our cuboctahedron number,

$$\boldsymbol{CO_n=C_n+C_{n-1}+4Pyr_{n-1}},\ \textbf{[13.1]}$$

or, substituting from **[8.3]**:

$$\boldsymbol{CO_n=C_n+C_{n-1}+Tet_{2(n-1)}},\ \textbf{[13.2]}$$

or, in standard algebra:

$$CO_n=\frac{10n^3-15n^2+11n-3}{3}.$$

From the section on the centred cube, we can also write:

$$\boldsymbol{CO_n}=CC_n+4Pyr_{n-1}=\boldsymbol{CC_n+Tet_{2(n-1)}}.\ \textbf{[13.3]}$$

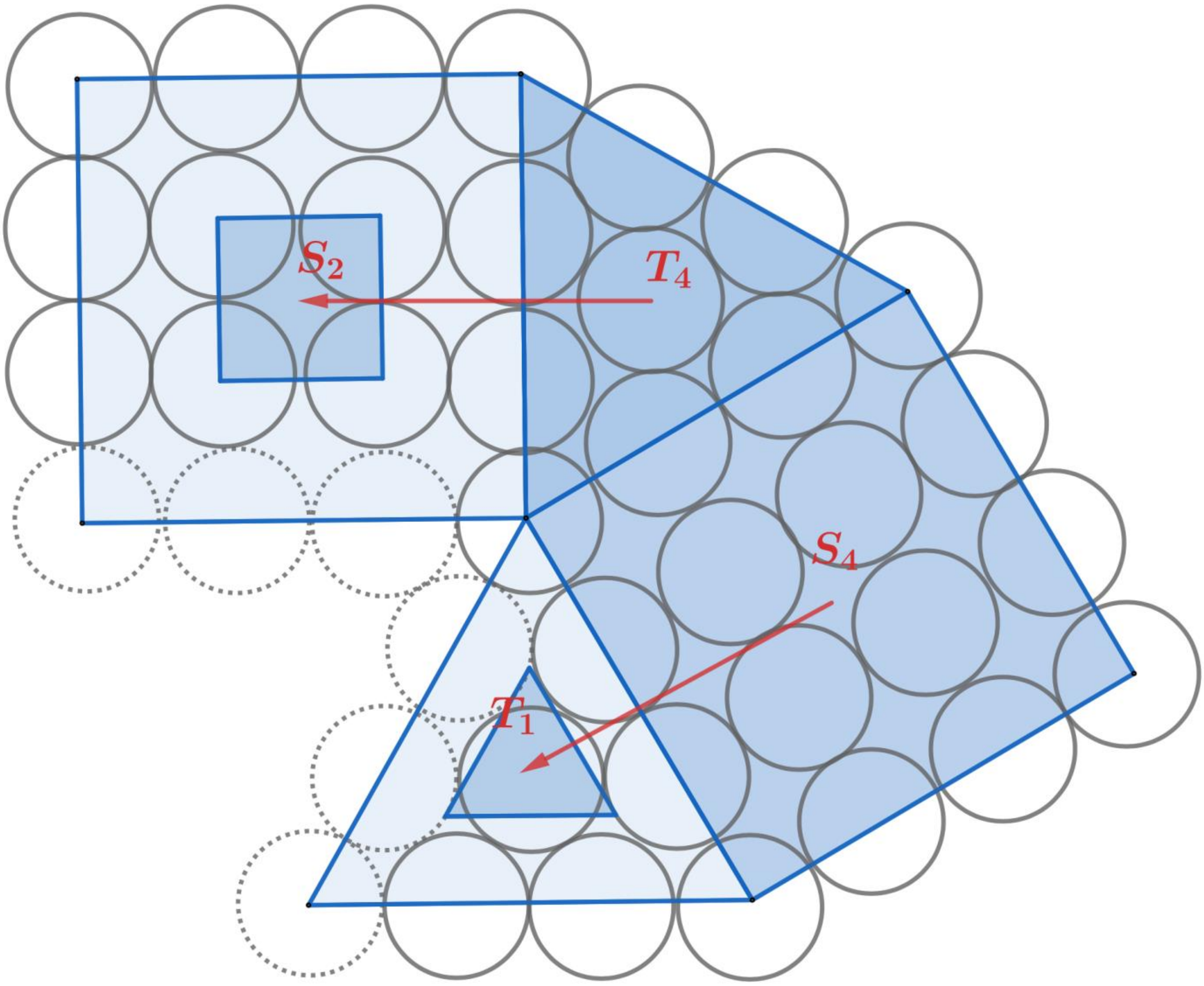
This formula may be confirmed by the method of finite differences.

The gnomon to the cuboctahedron encases it completely. The figure below shows part of a net of the shell. The dotted spheres coincide in pairs on an edge. Each of 8 triangular faces shares 3 vertices with others but one vertex is shared by two triangles so we must subtract $\frac{3\times 8}{2}$ units from the triangle totals. The triangle edges must be removed from the 6 squares, which reduces them by two size generations (see upper red arrow in figure). Thus we have:

$$CO_n = CO_{n-1} + 2(4T_n - 6 + 3S_{n-2}).\ \textbf{[13.4]}$$

By symmetry we can reverse the roles of square and triangle but must reduce the triangle generation by 3 (see lower red arrow in figure), leading to:

$$\boldsymbol{CO_n = CO_{n-1} + 2(3S_n - 6 + 4T_{n-3}).}\ \textbf{[13.5]}$$



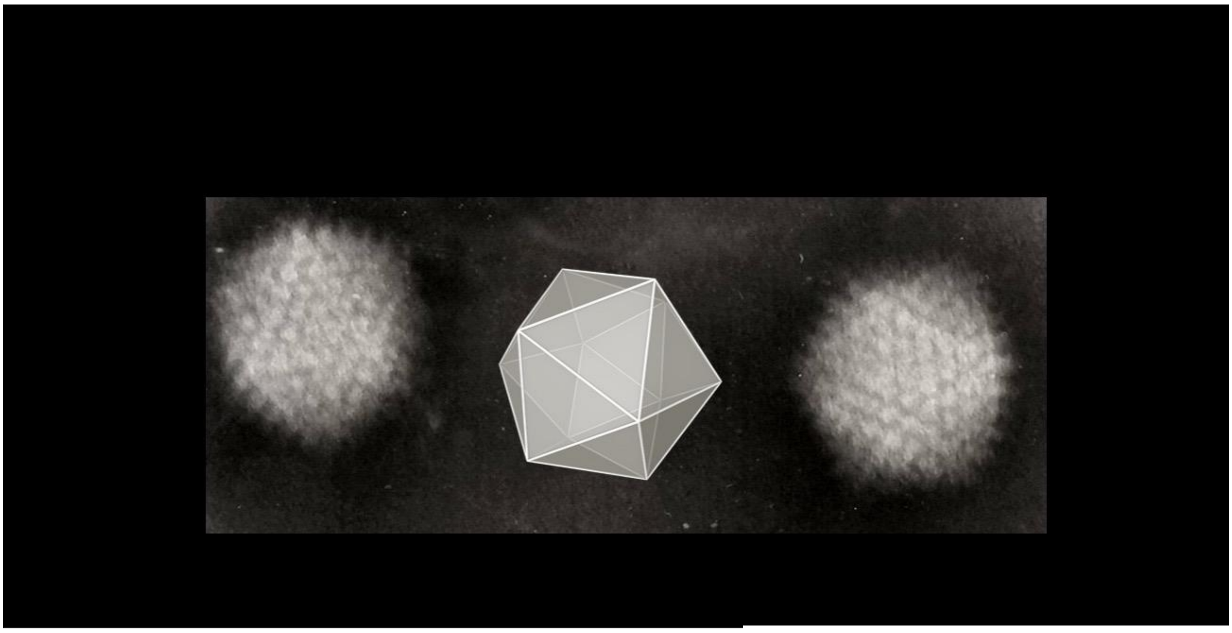
We can grow CO_n from the central unit, representing CO_1 by adding the gnomonic shells. The triangles become tetrahedra and the squares become pyramids. Using **[13.5]**, we have:

$$\boldsymbol{CO_n = CO_1 + 8(Tet_n - Tet_1) - 12(n-1) + 6Pyr_{n-2}.}\ \textbf{[13.6]}$$

We would like to produce the cuboctahedron by truncating the vertices of an octahedron or, equivalently, the vertices of a cube. Unfortunately, the cube and octahedron use a cubic packing of spheres, not the close packing we have used for the cuboctahedron.

Chapter 14

The Icosahedron Shell, I_n

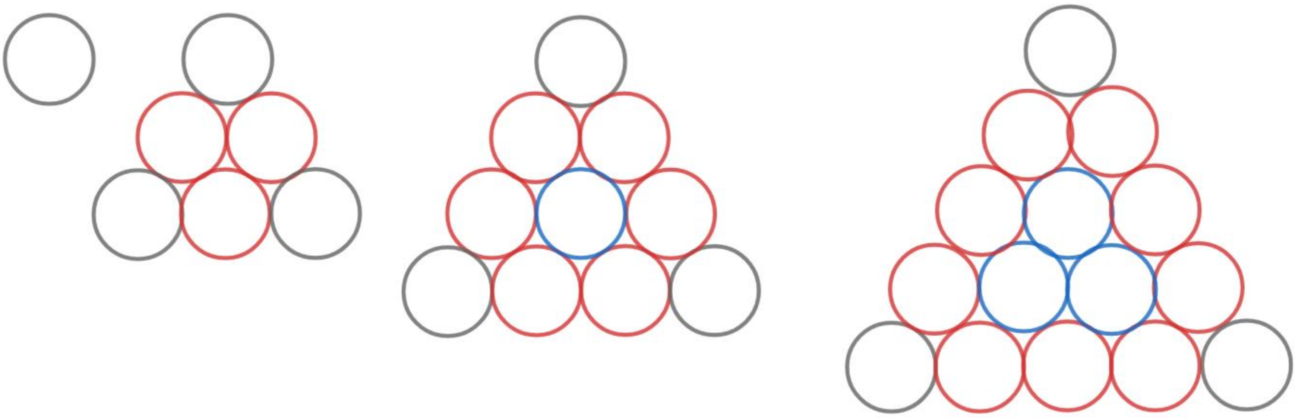


Structure of icosahedral adenovirus. Electron micrograph with an illustration to show shape

Source: Wikipedia

Our last example is not dissected from a packing but we take it because of a significant application.

Capsomeres are the protein units in the surface of a virus. Many viruses take the form of a regular icosahedron. The capsomeres are not usually spherical in form but we shall take them to be spheres in order to define an *icosahedron shell number*, I_n . (There is a simple way to calculate the number of *true* capsomeres due to Caspar & Klug. Go to www.magicmathworks.org , choose ‘Maths club projects’, then ‘The Scottish bubble and the Irish bubble’, and read chapter 1, ‘Thomson & Tammes’.) In our simplified structure, the sequence of sphere arrangements in an icosahedron face goes as follows and the sphere count runs as shown beneath.



1 per vertex

1 per vertex + 1 per edge

1 per vertex + 2 per edge + 1 per face

1 per vertex + 3 per edge + 3 per face

$I_1 = 1(12) = 12$

$I_2 = 1(12) + 1(30) = 42$

$I_3 = 1(12) + 2(30) + 1(20) = 92$

$I_4 = 1(12) + 3(30) + 3(20) = 162$

Note that the coefficient of the last bracket is a triangle number.

Applying the method of finite differences, we find that $\boldsymbol{I_n = 10S_n + 2}$. **[14.1]**

Unfortunately, because the shells are not dissected from a packing, they do not nest. There is therefore no gnomonic relationship between them.

Table of values

We have matched by colour, numbers which occur in more than one set (other than L_n & O_n). Where some mathematical interest attaches to this coincidence we note it below.

L_n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
O_n	1	3	5	7	9	11	13	15	17	19	21	23	25	27
T_n	1	3	6	10	15	21	28	36	45	55	66	78	91	105
S_n	1	4	9	16	25	36	49	64	81	100	121	144	169	196
CS_n	1	5	13	25	41	61	85	113	145	181	221	265	313	365
CH_n	1	7	19	37	61	91	127	169	217	271	331	397	469	547
Tet_n	1	4	10	20	35	56	84	120	165	220	286	364	455	560
Pyr_n	1	5	14	30	55	91	140	204	285	385	506	650	819	1015
Oct_n	1	6	19	44	85	146	231	344	489	671	891	1156	1469	1834
C_n	1	8	27	64	125	216	343	512	729	1000	1331	1728	2197	2744
CC_n	1	9	35	91	189	341	559	855	1241	1729	2331	3059	3925	4941
CO_n	1	13	55	147	309	561	923	1415	2057	2869	3871	5083	6525	8217

9
$CC_2 = S_3$.
To find a centred cube which is a square, the simplest case to examine has the two brackets in the defining expression equal: $2n - 1 = n^2 - n + 1$, an equation which yields the two values in our table, $n = 1, n = 2$.

10
$T_4 = Tet_3$.
The equality $T_a = Tet_b$ occurs frequently. It represents a redistribution of prime factors among three terms: $3a(a+1) = b(b+1)(b+2)$. For example, $T_{15} = Tet_8$: $(3)(3 \times 5)(2^4) = (2^3)(3^2)(2 \times 5)$.

19:
$CH_3 = Oct_3$.
The equation $CH_n = Oct_n$, i.e. the cubic $2n^3 - 9n^2 + 10n - 3 = 0$, factorises as $(n-1)(n-3)(2n-1)$, giving 1 and 3 as integer solutions.

25:
$CS_4 = S_5$.
This result represents the unique Pythagorean triple where the legs are consecutive.

36:
$S_6 = T_8$.
The lowest square triangle number >1. (They are infinite in number.)

55:
$Pyr_5 = T_{10}$.
Pyr_n and T_{2n} share the factor $n(2n+1)$. The resulting linear equation yields the result $n = 5$, which is therefore unique.

T_{10} is also a sum of consecutive triangles: $T_{10} = T_2 + T_3 + T_4 + T_5 + T_6$.

61:
$CH_5 = CS_6$.
The equation $CH_{n-1} = CS_n$, i.e. the quadratic $n^2 - 7n + 6 = 0$, factorises as $(n-1)(n-6) = 0$, giving solutions 1 and 6. The more general equation $CH_a = CS_b$ simplifies to $3T_{a-1} = 2T_{b-1}$, identifying two triangle numbers in a ratio of small numbers.

91:
$CH_6 = Pyr_6$.
The equation $CH_n = Pyr_n$, i.e. the cubic $2n^3 - 15n^2 + 19n - 6 = 0$, factorises as $(n-1)(n-6)(2n-1) = 0$, giving 1 and 6 as integer solutions.

$T_{13} = Pyr_6$.
The equation $T_{2n+1} = Pyr_n$ has the unique solution $n = 6$. (T_{2n+1} and Pyr_n share the factor $(n+1)(2n+1)$.)

Combining those two results, we see without having to solve a new equation, that $n = 6$ is the only positive integer for which the equation $T_{2n+1} = CH_n$ holds.

As we proceed along the natural number sequence, extracting the squares as they occur and also the runs of numbers in between which sum to squares, we find we have accumulated exactly 5 squares on reaching 10 and exactly 6 on reaching 13, hence the equalities we have met under the headings '55' and '91':

									T_{10}			T_{13}
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
					16				**25**			**36**
									Pyr_5			Pyr_6

169:
$CH_8 = S_{13}$,
$6T_7 + 1 = 8T_6 + 1$,
$3T_7 = 4T_6$, again identifying two triangle numbers in a ratio of small numbers.

There is a simple relation between the ratios of consecutive pairs of triangle numbers. Writing out the formula for each triangle number in each ratio $\frac{T_{k+1}}{T_k}$ and cancelling, we have:

$$\ldots \quad T_{r-2} \qquad T_{r-1} \qquad T_r \qquad T_{r+1} \qquad T_{r+2} \quad \ldots$$
$$\ldots \qquad \frac{r}{r-2} \qquad \frac{r+1}{r-1} \qquad \frac{r+2}{r} \qquad \frac{r+3}{r+1} \quad \ldots$$

Adding numerator and denominator of the second red fraction and putting the sum over the same sum for the first red fraction, we have the blue fraction in between: $\frac{(r+3)+(r+1)}{(r+1)+(r-1)} = \frac{2r+4}{2r} = \frac{r+2}{r}$.

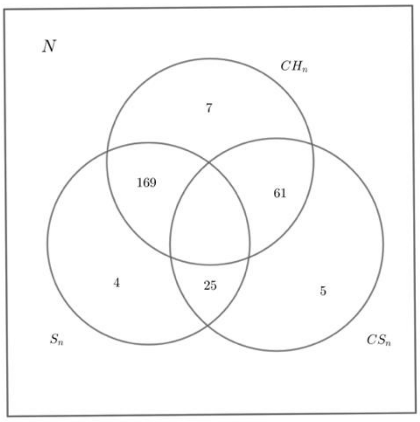
365
As mentioned in the text, $CS_{14} = S_{10} + S_{11} + S_{12}$, a sum of consecutive squares. Though this example is unique in featuring 5 consecutive squares, there is an infinite number of centred square numbers which

are sums of 3 consecutive squares. The next biggest is CS_{134}. $CS_{134} = S_{108} + S_{109} + S_{110}$. It turns out that, if $(y_n)^2$ is the middle of the three squares, and the CS suffix is $\frac{3(x_n+1)}{2}$, values are given by this recurrence relation:

$$x_n = 5x_{n-1} + 4y_{n-1}$$
$$y_n = 6x_{n-1} + 5y_{n-1}$$

1729
The most celebrated centred cube, the number of Hardy's taxicab on the occasion of a visit to the ailing Ramanujan, who pointed out that it is the smallest number which can be partitioned into a pair of cubes in two ways: $9^3 + 10^3$ and $1^3 + 12^3$.

With the following Venn diagram we can answer in the negative the question 'Can a number be a square, a centred square and a centred hexagon?'.



25 is the only square which is the sum of two consecutive squares. It is not a centred hexagon number. The central region must therefore remain empty.

Following our review of the different number types, we are in a position to express the same relationship in several different ways.

$$\begin{array}{lllllll} T_5 + T_6 & = Tet_6 - Tet_4 & = (T_3)^2 & = T_8 & = S_6 & = Pyr_6 - Pyr_5 & = 1^3 + 2^3 + 3^3, \\ T_5 + T_6 + T_7 + T_8 & = Tet_8 - Tet_4 & = (T_4)^2 & & = S_{10} & = Pyr_{10} - Pyr_9 & = 1^3 + 2^3 + 3^3 + 4^3. \end{array}$$

We can show that the second equality is the only example of four consecutive triangles summing to a square. We have:

$$(T_a + T_{a+1}) + (T_{a+2} + T_{a+3}) = S_c,$$
$$S_{a+1} + S_{a+3} = S_c\ ,$$

corresponding to the Pythagorean triple $(a, a + 2, c)$. Because the two legs have the same parity, they must represent a primitive triple scaled. Furthermore, they must both be even. Dividing by 2, we have $(p, p + 1, c)$. The only triple in which the legs differ by 1 is (3, 4, 5). Our case is therefore unique.

A note on inequalities:

In chapter **3** we have: $S_n \ < \ 2T_n \ < \ S_{n+1}$.

Differences: n $\quad n+1$

By analogy: $C_n \ < \ 6Tet_n \ < \ C_{n+1}$.

Differences: n(3n + 2) $\quad n+1$

Note in passing that $S_{n+1} - 2T_n = C_{n+1} - 6Tet_n$.

We can extend the inequality:

$C_{n+1} \quad < \quad 6Pyr_n$.

Difference: $n^3 - 2n - 1$

Note that, for $n > 5$, an increasing number of cubes fall between C_{n+1} and $6Pyr_n$.

Lists of identities

In the Table 1 and Table 2 terms to the left of the ‘=’ sign are to be added. In the cells shaded blue the identities are what we may call *figurate-homogeneous*: the figurate number type is of just one kind. Green-shaded cells show a simple gnomonic relationship (type **(b)** in the introduction, ‘simple’ meaning of the first order, a parent, not a grandparent or great grandparent). Orange-shaded cells show a relationship of the kind $P_{n-1} + P_n = Q_n$ (type **(a)** in the introduction).

Table 1: Identities involving 2-dimensional figurate numbers only

1	***L***	***O***	***T***	***S***	**GG**	**Tr**	$\boldsymbol{CS}$	***CH***	$=$	**Sum**	**No.**
1	L_{n-1}									L_n	1.1
2		O_{n-1}								O_n	2.1
	$L_{n-1} + L_n$									O_n	2.2
	L_n		T_{n-1}							T_n	3.1
						$Tr_{m-1,n} + Tr_{m,n}$				$TTr_{m,n}$	3.2
		O_n	T_{n-2}							T_n	3.3
	$3L_{n-1}$		T_{n-3}							T_n	3.4
			$T_{n-1} + T_n$							S_n	3.5
1			$8T_n$							S_{2n+1}	3.6
			$T_{n-1} + T_{n+1}$							$2T_n + 1$	3.7
			$3T_n + T_{n+1}$							T_{2n+1}	3.8
			$T_{n-1} + 3T_n$							T_{2n}	3.9
				S_tS_n						S_{tn}	3.10
	L_n			S_n						$2T_n$	3.11
			$T_{S_n} - T_{S_n-1}$							S_n	3.12
			$T_{n-1} + 6T_n + T_{n+1}$							S_{2n+1}	3.13
						$3Tr_{2n,n}$				T_{3n}	3.14
			$3(T_{2n} - T_n)$							T_{3n}	3.15
			$3T_n$	$3S_n$						T_{3n}	3.16
			$3(T_{n-1} + 2T_n)$							T_{3n}	3.17
			$T_{3(n-1)}$			$3Tr_{3n-1,3n-2}$				T_{3n}	3.18
			$T_{3(n-1)} + 3 (T_{3n-1} - T_{3n-2})$							T_{3n}	3.19
1						$3Tr_{2n+1,n+1}$				T_{3n+1}	3.20
1			$3(T_{2n+1} - T_{n+1})$							T_{3n+1}	3.21
-2			$3T_{n+1}$	$3S_n$						T_{3n+1}	3.22
-2			$3(T_{n-1} + T_n + T_{n+1})$							T_{3n+1} (Compare 6.3)	3.23

			T_{3n-2}			$3Tr_{3n,3n-1}$				T_{3n+1}	3.24
			$T_{3n-2} + 3(T_{3n} - T_{3n-1})$							T_{3n+1}	3.25
						$3Tr_{2n+1,n}$				T_{3n+2}	3.26
			$3(T_{2n+1} - -T_n)$							T_{3n+2}	3.27
			$3T_n$	$3S_{n+1}$						T_{3n+2}	3.28
			$3(2T_n + T_{n+1})$							T_{3n+2}	3.29
			T_{3n-1}			$3Tr_{3n+1,3n}$				T_{3n+2}	3.30
			$T_{3n-1} + 3(T_{3n+1} - T_{3n})$							T_{3n+2}	3.31
		O_n		S_{n-1}						S_n	4.1
					$GG_{l,m} + GG_{m,n}$					$GG_{l,n}$	4.2
	$4L_{n-1}$			S_{n-2}						S_n	4.3
		$3O_{n-1}$		S_{n-3}						S_n	4.4
										See text	4.5
			T_{n-2} $+2T_{n-1}$ $+T_n$							CS_n	5.1
1			$4T_{n-1}$							CS_n	5.2
			$4T_{n-1}$				CS_n			S_{2n-1}	5.3
										See text	5.4
-1	L_n						CS_n			T_{2n-1}	5.5
			$8T_{2n-1}$				CS_{n+1}			CS_{3n}	5.6
-1				S_{4n-1}			CS_{n+1}			CS_{3n}	5.7
			$T_{n+1} + 2T_{2n+1}$							T_{3n+2}	5.8
										See text	5.9
1				S_{2n-1}						$2CS_n$	5.10
			$12T_n$					$7CH_{n+1}$		CH_{3n+2}	6.1
1			$6T_n$							CH_{n+1}	6.2
1			$9T_n$							T_{3n+1}	6.3
										See text	6.4
			$3(2T_n + T_{n+1})$							T_{3n+2}	6.5
			$T_{k+1}T_n + T_kT_{n+1}$							$T_{kn+(k+n)}$	6.6
			$(T_{n-1})^2 + (T_n)^2$							T_{S_n}	6.7
			$T_{a-1}T_{b-1} + T_aT_b$							T_{ab}	6.8
			$(T_n)^4$- $(T_{n-1})^4$							$T_{S_n}C_n$	6.9
			T_n	S_{2n+1}						T_{3n+1}	6.10
			$2T_{n-1}$				CS_n			CH_n	6.11
				S_{2n-1}			CS_n			$2CH_n$	6.12
			$T_{n-2} + 4T_{n-1} + T_n$							CH_n	6.13
			$3T_{n-1}$					CH_n		T_{3n-1}	6.14

Table 2: identities involving 3-dimensional figurate numbers

1	1-D figure (× factor)	2-D figure (× factor)	3-D figure	Tet_n	Pyr_n	Oct_n	C_n	CC_n	CO_n	=	Sum	No.
		T_n		Tet_{n-1}							Tet_n	7.1
		S_n		Tet_{n-2}							Tet_n	7.2
1		$3T_{n-1}$		Tet_{n-3}							Tet_n	7.3
		S_n			Pyr_{n-1}						Pyr_n	8.1
				$Tet_{n-1} + Tet_n$							Pyr_n	8.2
				Tet_{2n}							$4Pyr_n$	8.3
											See text	8.4
											See text	8.5
					$Pyr_{n-1} + Pyr_n$						Oct_n	9.1
		S_n			$2Pyr_{n-1}$						Oct_n	9.2
	L_n			$4Tet_{n-1}$							Oct_n	9.3
		CS_n				Oct_{n-1}					Oct_n	9.4
											See text	9.5
											See text	9.6
				$4Tet_n$		Oct_{n+1}					Tet_{2n+1}	9.7
	L_{n+1}			$8Tet_n$							Tet_{2n+1}	9.8
		CH_n					C_{n-1}				C_n	10.1
1		$6T_{n-1}$					C_{n-1}				C_n	10.2
	L_n			$6Tet_{n-1}$							C_n	10.3
2		$6S_{n-1}$					C_{n-2}				C_n	10.4
											See text	10.5
	O_n				$24\,Pyr_{n-1}$						C_{O_n}	10.6
	O_n		$8O_nT_{n-1}$								C_{O_n}	10.7
				$2Tet_{n-1}$		Oct_n					C_n	10.8
				$2Tet_{n-1}$	$Pyr_{n-1} + Pyr_n$						C_n	10.9
				$Tet_{n-2} + 4Tet_{n-1} + Tet_n$							C_n	10.10
		S_n		$2Tet_{n-1}$	$2Pyr_{n-1}$						C_n	10.11
											See text	11.1
											See text	11.2
											See text	11.3
			$O_nT_{n-2} + O_nT_n$								CC_n	12.1
											See text	12.2
					$18Pyr_{n-1}$			CC_n			C_{2n-1}	12.3
					$2Pyr_{n-1}$	$Oct_{n-1} + Oct_n$					CC_n	12.4
		$18(S_{n+1} + 1)$						CC_n			CC_{n+3}	12.5
		$T_{2(n-1)}$						CC_n			O_nS_n	12.6
											See text	13.1
											See text	13.2
				$Tet_{2(n-1)}$				CC_n			CO_n	13.3
											See text	13.4
											See text	13.5
											See text	13.6
2		$10S_n$									I_n	14.1