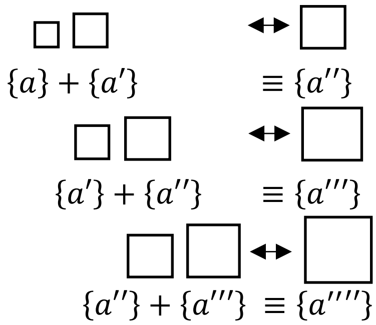
DISSECTION SEQUENCES

Dissection is the art of cutting up one shape and using the pieces to make another. In the dissection of plane figures the invariant is area. If the shape is a polygon, the number of pieces needed is always finite. The art lies in using the fewest. Over the years there have been many examples in Symmetry+ . The classic account is ‘Geometric Dissections’ (1972), in which Harry Lindgren reveals the techniques he used. Lindgren’s mantle passed to Greg N. Frederickson, whose ‘Dissections: plane and fancy’ (1997) charts the history of the art, extends Lindgren’s toolbox and displays the results. Like Lindgren and Frederickson, we shall be concerned only with regular polygons, (and in our case only with the convex, not the star, types.)

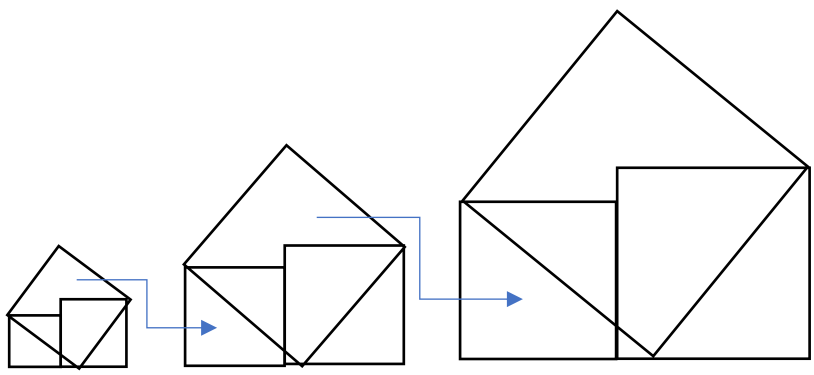
In this piece we extend the idea from single-stage to multi-stage dissections, sequences in which we *iterate* some basic operation. There are three types. We represent each schematically and symbolically then give examples. $\{a\}$ stands for a regular a-gon. $\{a\}, \{a'\}, \{a''\}$, ... are similar.

(1) Step shift sequences

Here the polygons we start with are not congruent but are similar.



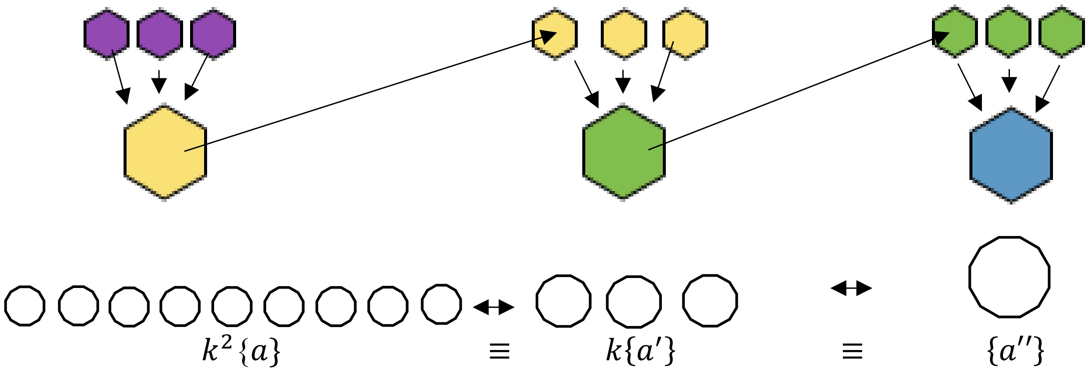
In the following example, we take a standard Pythagoras dissection through three iterations - but could continue ad infinitum. A particular ratio is chosen for edge length of larger square to edge length of smaller square in each pair, in my case 4:3. Extra pieces are introduced at each stage till, in the final stage, all the pieces are in place. I managed with 11. All the pieces are based on the 3:4:5 right triangle. 6 are similar versions of it; the other 5 are either trapezia which complete such triangles or composites of such triangles and trapezia. *Try to work out the shapes of the pieces needed and cut them out from thick card.*



Notice that the sloping square is made by translating the bottom left triangle to a position top right, and the bottom right triangle to a position top left.

(2) Dissection hierarchies

The idea here is to turn k congruent polygons into one, then k of *those* into one, then k of *those* into one, ... ad infinitum:

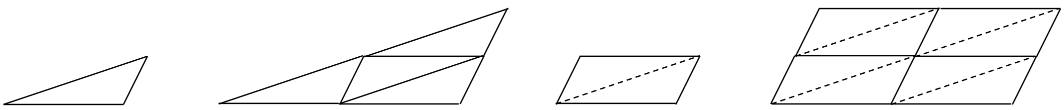


Our plan is to create infinite trees in which we *iterate* the replication process in a hierarchical manner. Before listing examples, I bring together some simple principles which are scattered over many past issues of Symmetry+.

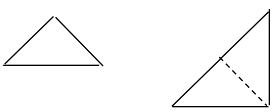
We find that all the polygons to be found in regular and semiregular tessellations: the equilateral triangle, the square, the regular hexagon, octagon and dodecagon, can be dissected down the generations if cut into subsets of just 4 shapes: the 30° and 45° rhombuses and the $30° - 60°$ and $45° - 45°$ right triangles.

What do these 4 shapes have in common?

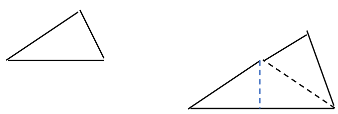
Following Solomon W. Golomb, who coined the term, they're all *rep-tiles:* we can make bigger, similar shapes by aggregating copies of the original. These are the RNA which make our breeding programme possible. This self-replicating property applies to all triangles, including therefore our two right-angled ones, and all parallelograms, including therefore our two rhombuses. As shown by the dotted lines below, the second fact follows from the first. The *replication numbers* in both cases are the squares: 4, 9, 16, … Accordingly we say that these shapes are 'rep-4', 'rep-9', 'rep-16' and so on.



The $45° - 45° - 90°$ triangle is special because it's also rep-2:



The $30° - 60° - 90°$ triangle is special because it's also rep-3:

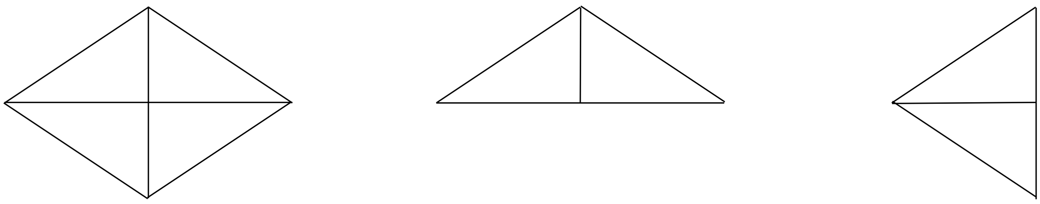


Our 4 elementary shapes contain only 4 non-obtuse angles. All are unit fractions of a half-turn. All stand in simple ratios to each other. (Note the top-right to bottom-left symmetry axis to the table.)

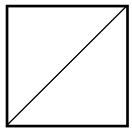
	30°	45°	60°	90°
30°		3:2	2:1	3:1
45°			4:3	2:1
60°				3:2
90°				

As a result, they may be arranged in many combinations round a vertex.

We can obtain four $30° - 60° - 90°$ triangles by slicing the 60° rhombus along both diagonals. Where this triangle appears in pairs in our dissections, it is joined either by its shortest side, to produce a $120°$ isosceles triangle, or by its middle side, to produce an equilateral triangle; and those pairs can each combine as the original rhombus:



In the same way, a pair of $45° - 45° - 90°$ triangles (which we can also call $90°$ isosceles triangles) often appear as the square from which they might have been dissected:



If a $90°$ isosceles triangle presents a short side to the boundary and we flip it over, we scale the polygon by $\sqrt{2}$ and the branching number of our dissection tree is 2. If we do the same with the $120°$ isosceles triangle, the factor is $\sqrt{3}$ and the branching number 3. These options give 3 families of edge length in terms of a unit: integers, integers $\times$ odd powers of $\sqrt{2}$ and integers $\times$ odd powers of $\sqrt{3}$. In our trees, then, the scale factor is $\sqrt{k}$, the *branching number* *k* and, if we take a horizontal section through a vertical tree, what is conserved is area.

We start at level 0 and descend the trees, level by level. The polygons on level 0, the set P_0 , comprising the *k* polygons $P_{0.1}, P_{0.2}, P_{0.3}, \dots, P_{0.k}$, may or may not all be dissected into the same set of pieces. But those on levels 1, 2, 3, … are.

Level: 0 Polygon: $P_{0.1}, P_{0.2}, P_{0.3}, \dots, P_{0.k}$

1 $P_{1.1}$ $P_{1.2}$ $P_{1.3}$, …., $P_{1.k}$

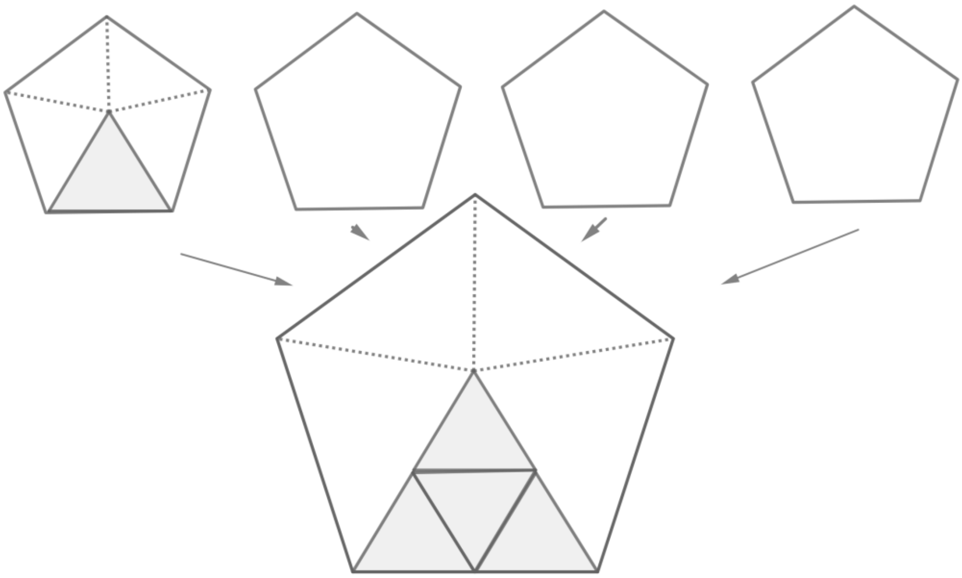
2 $P_{2.1}$

If the P_0 polygons are rep-k, we can always achieve an infinite hierarchy with this branching number. (As noted, $k = n^2$ takes care of all triangles and parallelograms.)

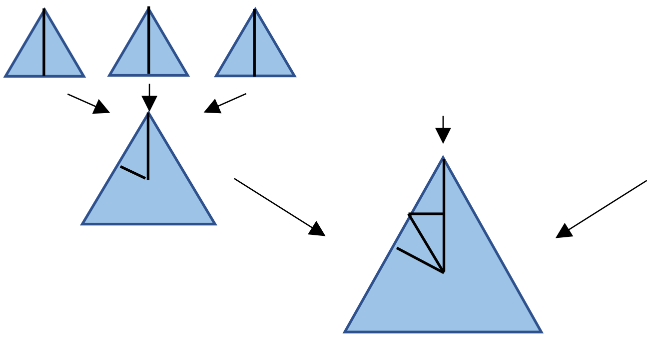
The same is true if the P_0 polygons are not rep-tiles themselves but can be dissected into them.

Where it is instructive, we shall give the *substitution rules* by which a larger polygon is obtained from a smaller one.

Because every triangle is a rep-tile of order s^2, and we can divide every regular n-gon into n isosceles triangles, we can produce an infinite tree with branching number $k = s^2$ for every regular polygon:

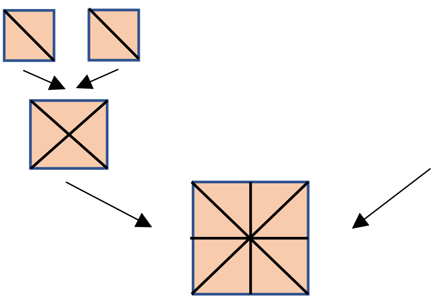


The equilateral triangle, *k* = 3



We split it into two $30° - 60° - 90°$ triangles. These are rep-3 so we can achieve a tree with this branching number. Accordingly, the number of P_0 tiles in successive generations run 6, 18, 54, ...

The square, *k* = 2

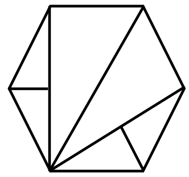


In the same way, we can split the square into two isosceles right triangles and take advantage of the fact that these are rep-2 to create an infinite tree with branching number 2. The tile numbers are powers of 2.

The hexagon, *k* = 3

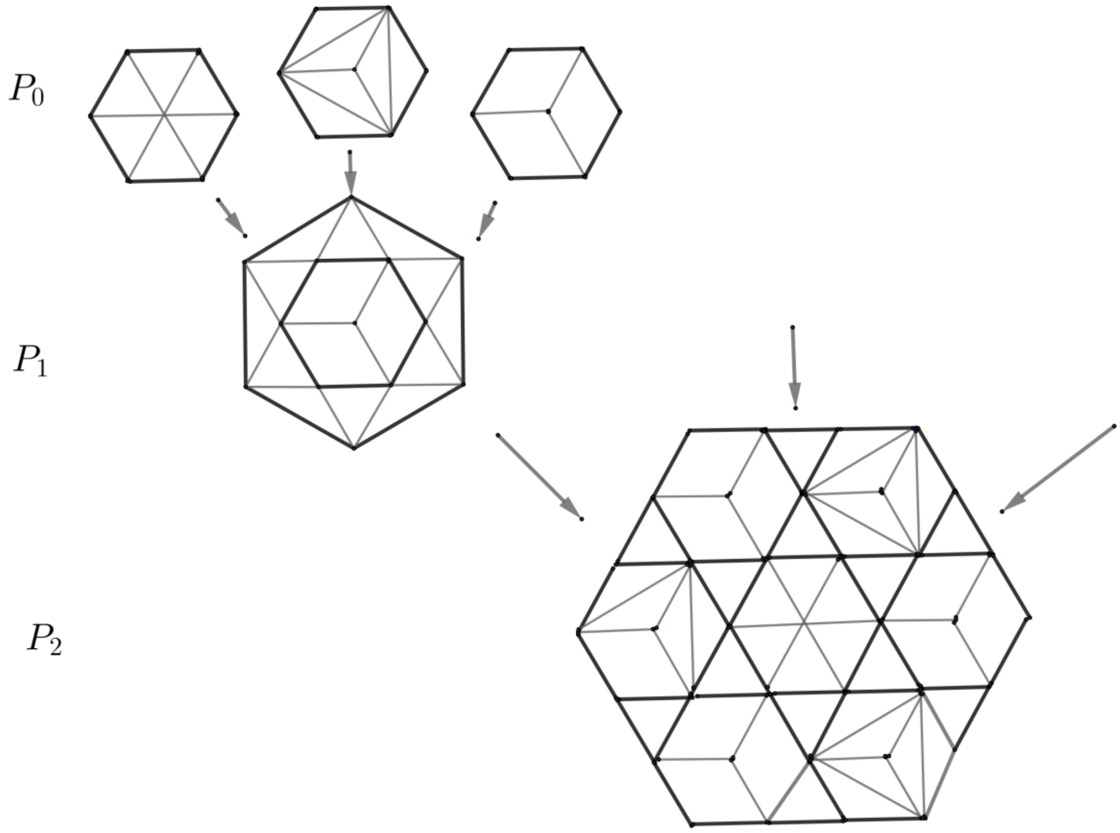
We give three dissections.

(**a**) The regular hexagon is not a rep-tile. We can always split it into $30° - 60° - 90°$ triangles as we did with the equilateral triangle. And we can reduce their number in this case by dissecting the hexagon into triangles of different sizes:



So we can achieve a branching number of $k = 3$. And the number of tiles runs 18, 54, 162, ...

(b) Consider this P_0 set of 15 tiles. Each is rep-n^2. But we find we can produce both the P_1 and P_2 sets with $k = 3$. This means that we can get to P_3 to P_5 to P_7, ... just by scaling P_1; and from P_4 to P_6 to P_8, ... just by scaling P_2. Thus, by this leapfrogging process, we can achieve all P_n to $n = \infty$.



The number of tiles runs 15, 45, 135, ... , a small improvement on the preceding.

We shall dignify the leapfrog process as a theorem.

The leapfrog theorem:

If we have k congruent polygons, $P_{0.1}, P_{0.2}, P_{0.3}, \ldots, P_{0.k}$, dissected into any set of triangles and parallelograms, and find we can assemble from those k a single similar polygon, $P_{1.1}$, and from k copies of this, $P_{1.1}, P_{1.2}, P_{1.3}, \ldots, P_{1.k}$, a similar polygon $P_{2.1}$, then we can assemble a *set P_n from k^{n-1} copies of the set P_0 for all values of n.*

Proof:

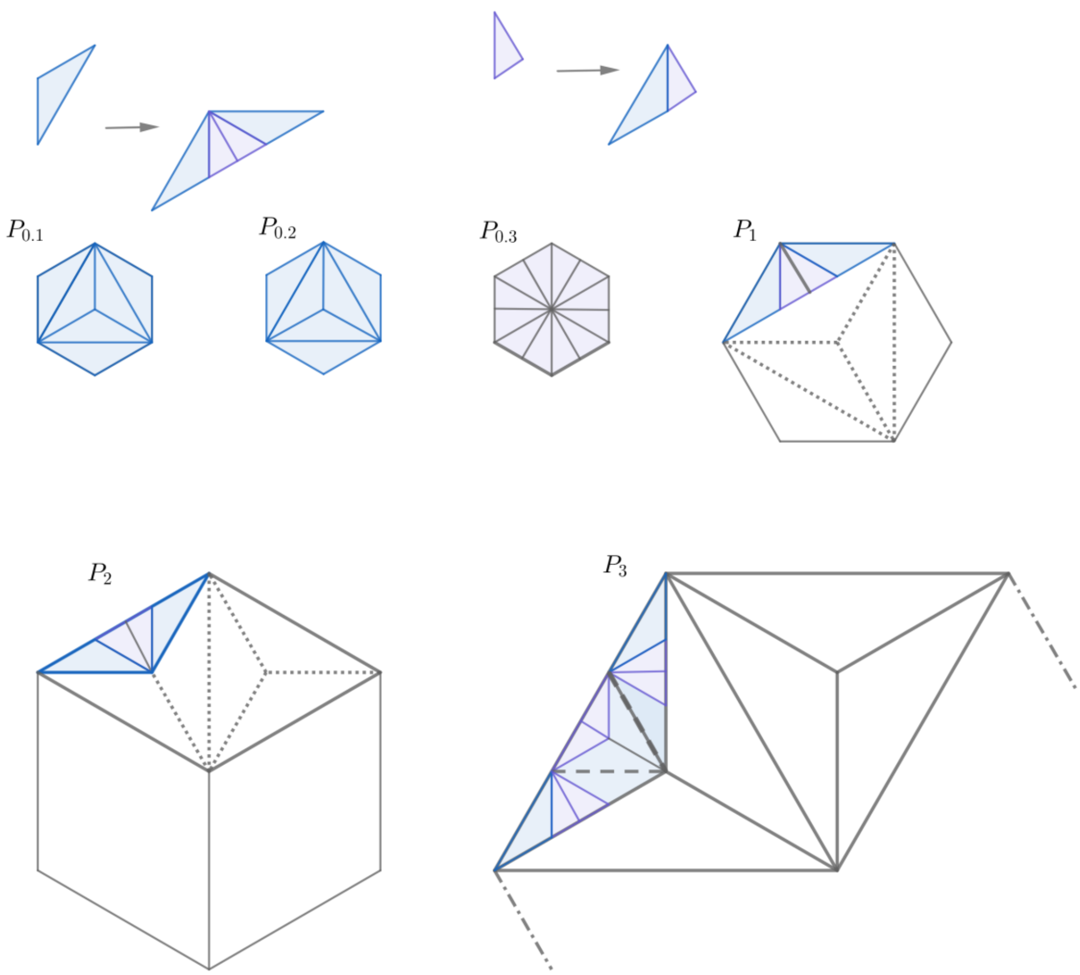
Any triangle, whence also any parallelogram, has the property that k^2 congruent copies can be assembled into a single similar shape. (It is a rep-tile.) Thus, by scaling, we can assemble from k^2 copies of set P_1 the set P_3, whence also $P_5, P_7, P_9, \ldots$. But we know we can assemble the set P_2 , hence by scaling, P_4 P_6 P_8, ... Thus we can assemble any P_n set from k^{n-1} copies of the P_0 set.

A corollary:

Where the P_0 set are dissected identically, we only need establish the P_1 set to ensure an infinite tree because we can take our first inductive step from P_0 to P_2.

In these cases the substitution rules relate alternate generations.

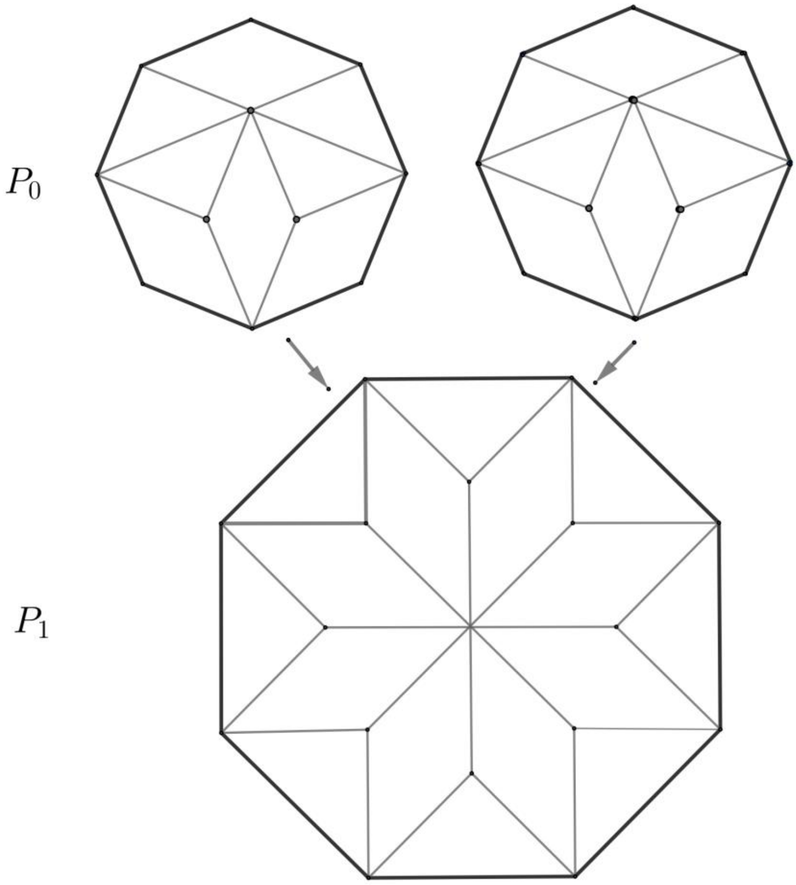
(c) The next dissection is interesting because of the substitution rules.



The octagon, *k* = 2

We give two alternatives.

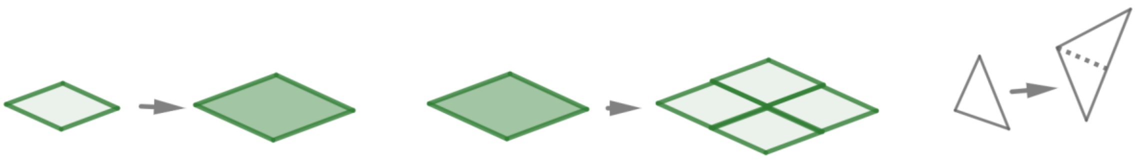
(a)

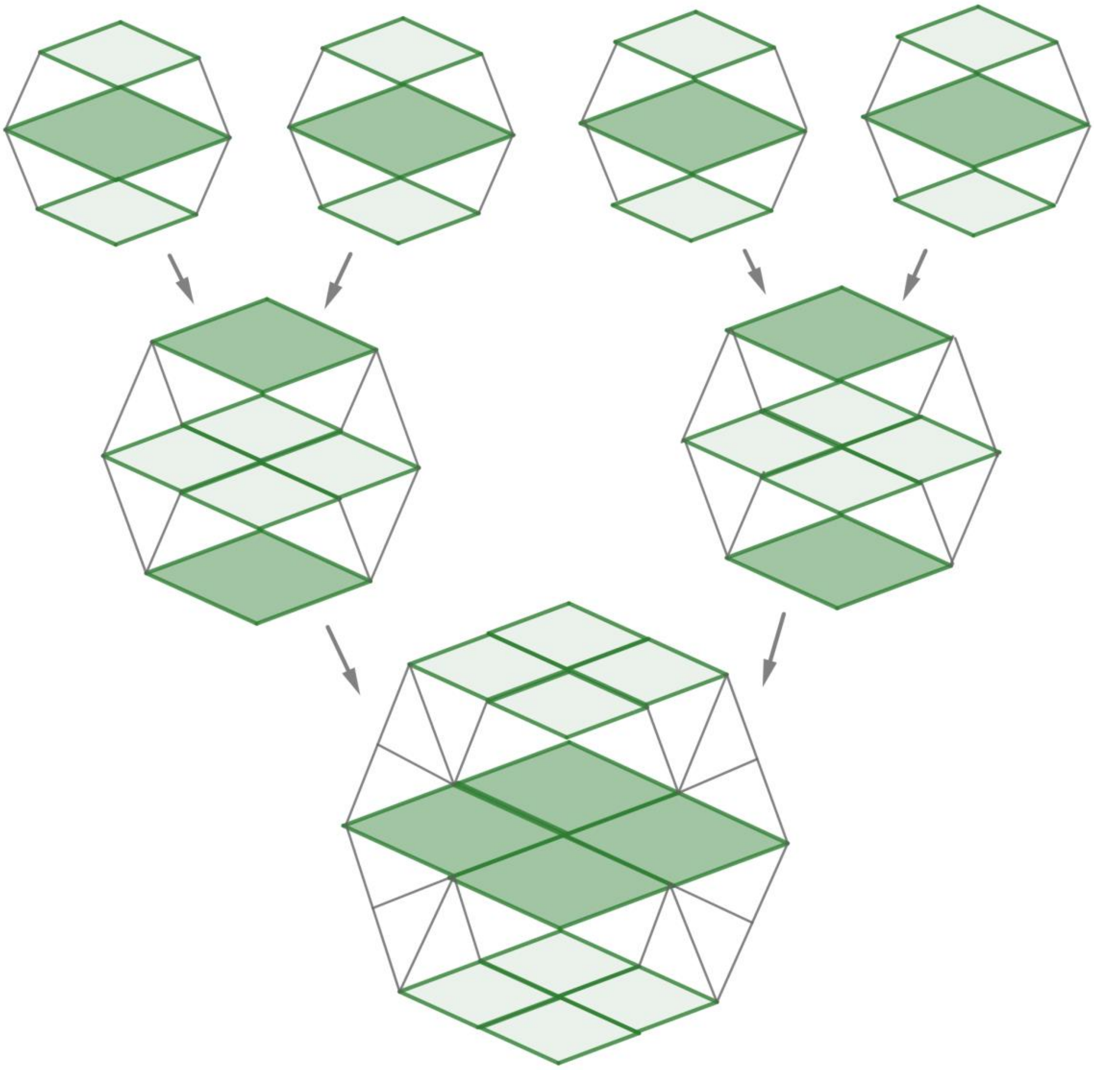


Again we can dissect the figure into rep-tiles, and we can achieve P_1. Since here the P_0 dissections are identical, we can invoke the corollary to the leapfrog theorem to ensure an infinite tree.

To draw a P_2 octagon, complete the squares by adding 8 more right isosceles triangles, and put 45° rhombuses in between.

(b) Here is an alternative. We first give the substitution rules. The number of pieces in each P_0 shape goes down from 8 to 7:

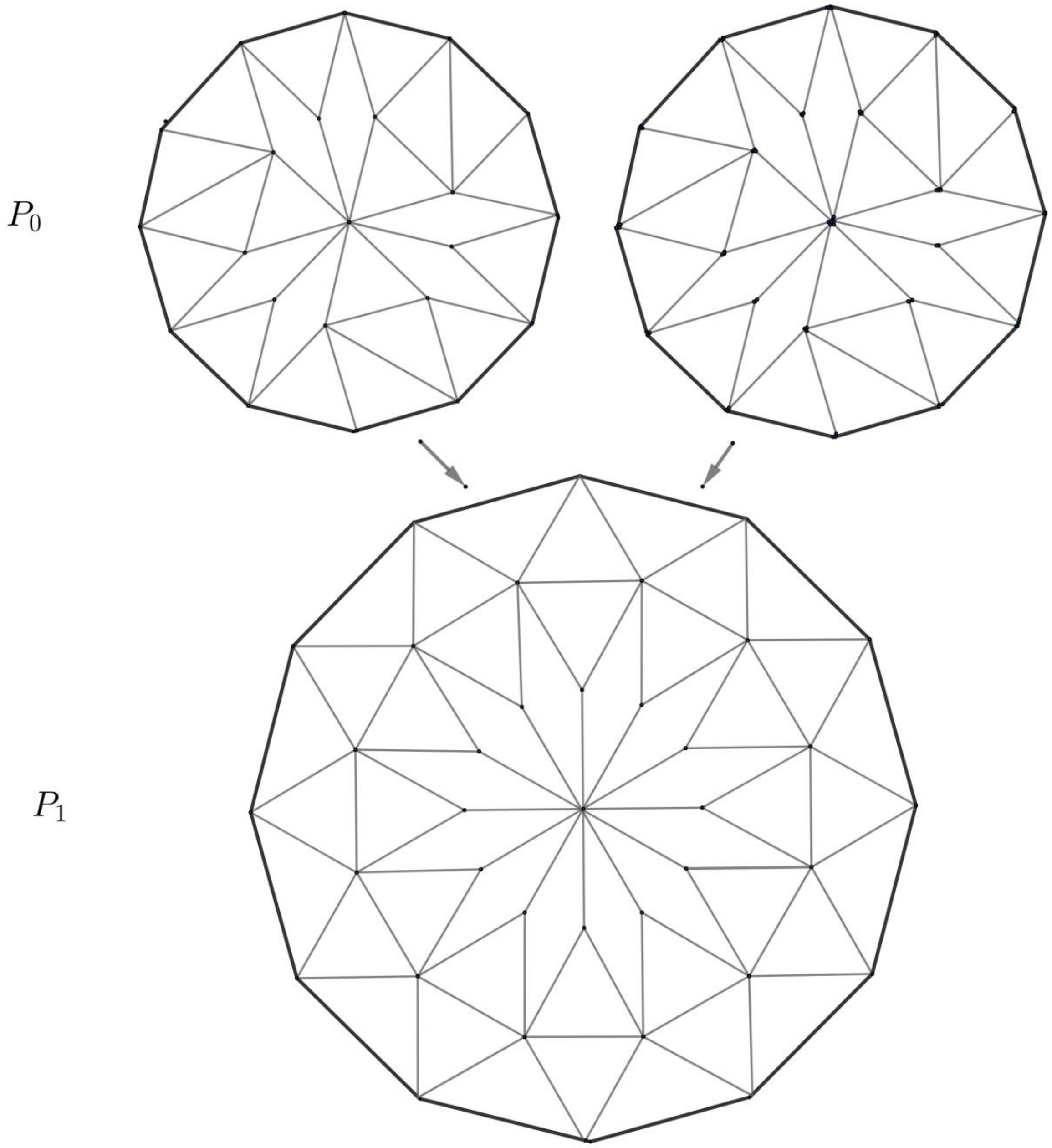




In that example the branching number (2) is not the same as the replication number for every one of the polygon's constituent rep-tiles: a rhombus is rep - 4. But the substitution rules explain this mismatch: the pale and dark green rhombuses alternate in position with the generations.

The dodecagon, *k* = 2

As for the octagon, so for the dodecagon.



Analyse that solution. Why, despite the triangle and rhombus only being rep–4, is a branching number of 2 possible? What is it about the interior angles of the constituent shapes which maximises the number of ways they can fit together?

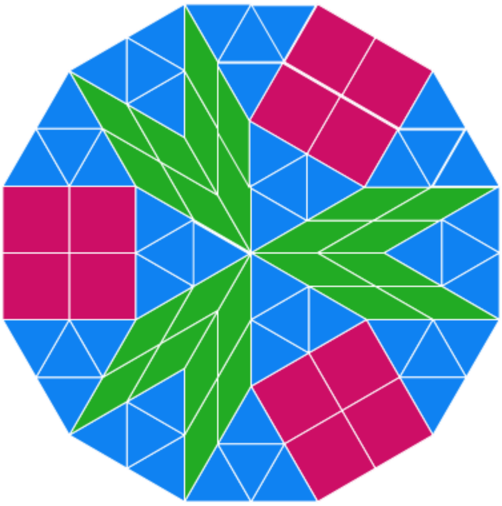
Beyond 12 sides?

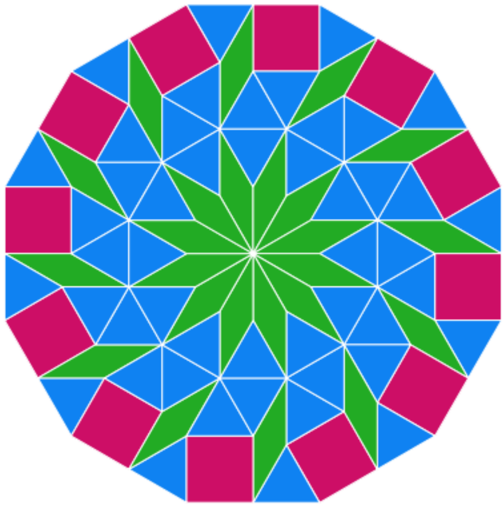
Investigate polygons with more than 12 sides.

The artistic dimension

Though we can happily aggregate square numbers of tiles in our dissections, where possible it is more satisfying to arrange them with an aesthetic end in view. Your aim might be to produce the greatest number of symmetry axes, preserve rotational symmetry, maintain a concentric pattern, or, as suggested above, embed smaller copies of the polygon in the larger versions. (These possibilities are not exclusive.)

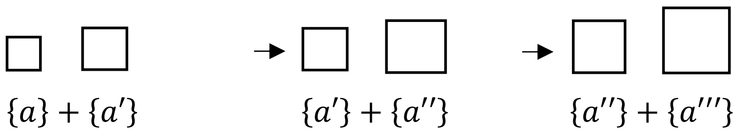
Go to mathigon.org/polypad#polygons, where you will find an interactive tile environment, and make a polygon of the P_2 set for the dodecagon. Below is my attempt. On the left is the simple scaling solution; on the right, a dissection with different symmetries. As you see, we have a generation 0 dodecagon in the centre (not the same as those of our set P_0) and rotation symmetry of order 12. You can obtain a different pattern by swapping a square and an equilateral triangle for an equilateral triangle and two 30° rhombuses.





(3) Rep-multiple tile sets

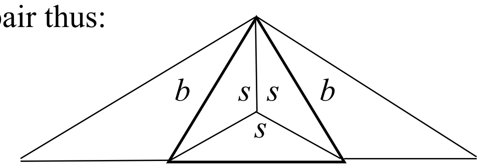
We can extend the idea of a rep-tile by recognising a *rep-multiple tile set* or *setiset*. This is a set of polygons with the property that a polygon similar to any member of the set can be made by combining all the members in a certain ratio. In this diagram the set consists of two, similar polygons:



The pair of 120° isosceles triangles

An example of such a set is the pair of 120° isosceles triangles with edges in the ratio $\sqrt{3} : 1$. We shall represent the smaller by the symbol *s*, the larger by the symbol *b*.

We can assemble a similar triangle from the pair thus:



This represents a scaling of *b* by $\sqrt{3}$, and therefore a scaling of *s* by 3.

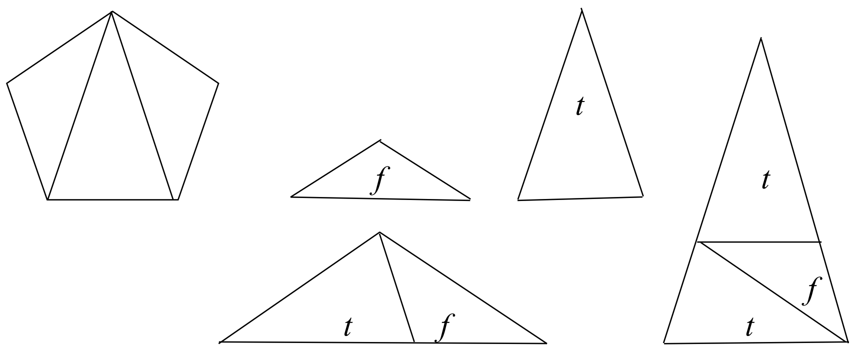
Beginning with our original triangles, *matrix multiplication* tells us how the enlarged triangles result from combining old ones on each iteration. A matrix is a rectangular array of numbers shown in a round bracket. The four brackets below are all matrices. The arrangement is a compact way of showing which numbers multiply and add to give us our new shapes. We've picked out a red number in the 2×2 matrix and shown which number it multiplies in the 2×1 matrix on its immediate right. Likewise with the blue numbers. And we've shown where the sum appears in the last bracket. *Study how the same operation with the second row of the square matrix leads to* b_{old} becoming s_{new}.

$$\begin{pmatrix} b_{new} \\ s_{new} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_{old} \\ s_{old} \end{pmatrix} = \begin{pmatrix} 2b_{old} + 3s_{old} \\ b_{old} \end{pmatrix}.$$

On each iteration, the triangles are scaled by $\sqrt{3}$.

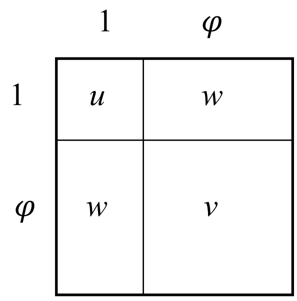
The pair of golden triangles

An example where the pair are not similar is the pair of isosceles triangles into which the diagonals divide the regular pentagon (the *golden* triangles).



Here, as you see from the figures in the second row, $\begin{pmatrix} t_{new} \\ f_{new} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t_{old} \\ f_{old} \end{pmatrix}$. The scale factor is the golden ratio $\varphi = \frac{\sqrt{5}+1}{2}$.

The golden square

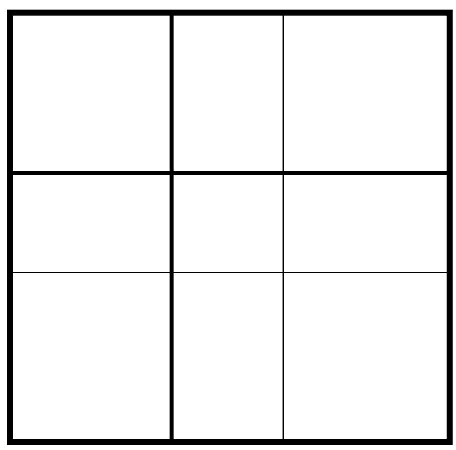


There can be more than two members in a set.

Here we dissect what we may call the *golden* square into three polygons: a golden rectangle and a pair of squares in golden ratio.

Try to complete this matrix equation. You can check your answer by seeing how the new shapes appear in the big square. *Notice how the old square nests in the new one.*

$$\begin{pmatrix} u_{new} \\ v_{new} \\ w_{new} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_{old} \\ v_{old} \\ w_{old} \end{pmatrix} = \begin{pmatrix} & + & \\ + & + & + \\ & + & + \end{pmatrix}.$$



The scale factor is again φ.

Write the area of the new square in two ways:
as the old one scaled up: $\varphi^2(\varphi + 1)^2$

and as the new one: $(2\varphi + 1)^2$.
Expand each expression using the handy fact that $\varphi^2 = \varphi + 1$.
Do you get the same answer?