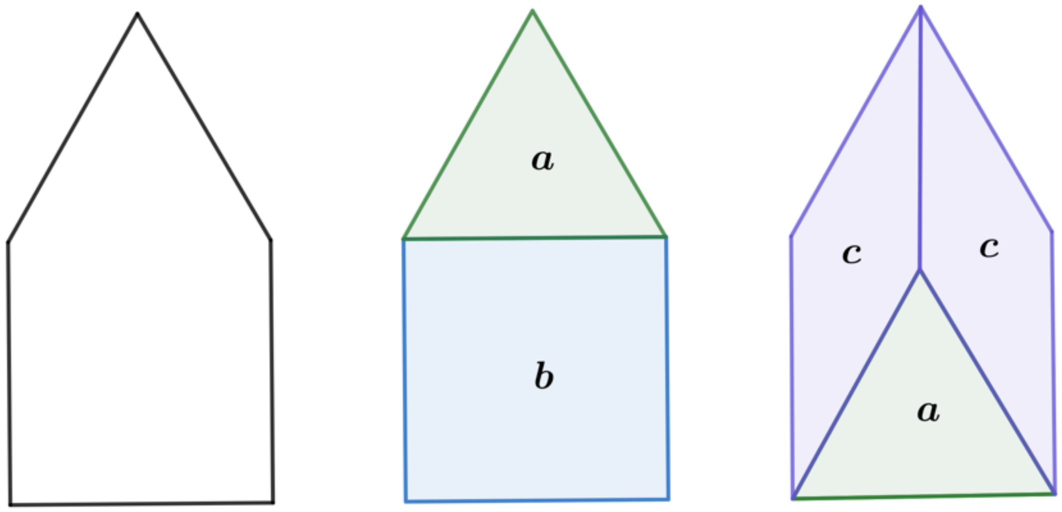
Dissection algebra

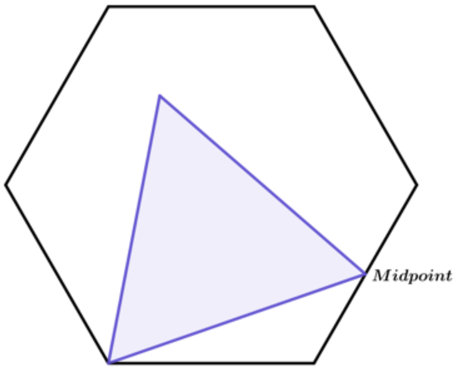
Dissection is the ancient art of cutting up one plane figure and rearranging the same pieces to make another, or filling the same outline with different pieces. Sometimes we can use this technique to avoid calculation. If we give each piece of the same shape and size the same letter, we can change geometry into algebra and write an equation. As soon as it enters the equation, our letter ceases to be a label and becomes a measure of area. Here are some examples.

1.



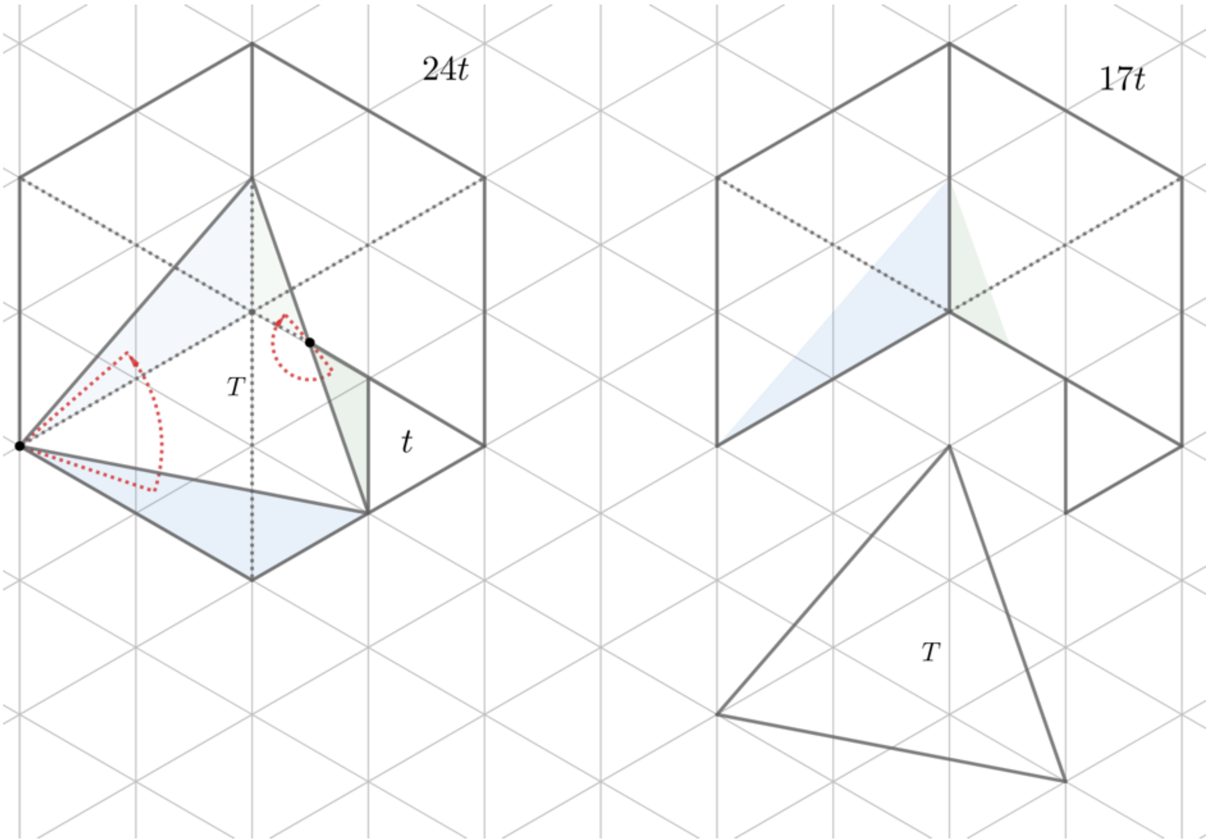
$$a + b = a + 2c.$$
$$b = 2c.$$

2.

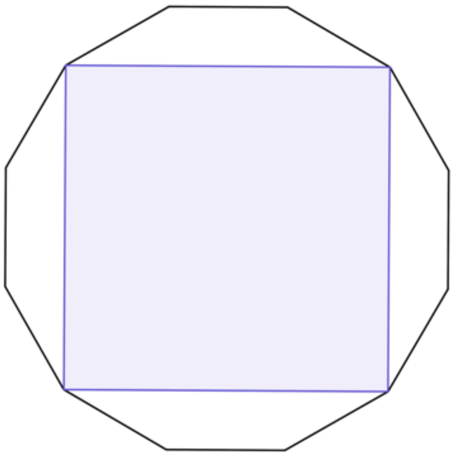


What fraction of the area of this regular hexagon is occupied by the equilateral triangle?

Below, we work in terms of the smallest constituent triangle. The area of the hexagon is $24t$. The dissection removes $17t$, leaving $T = 24t - 17t = 7t$, so the answer to the question is $^7/_{24}$.

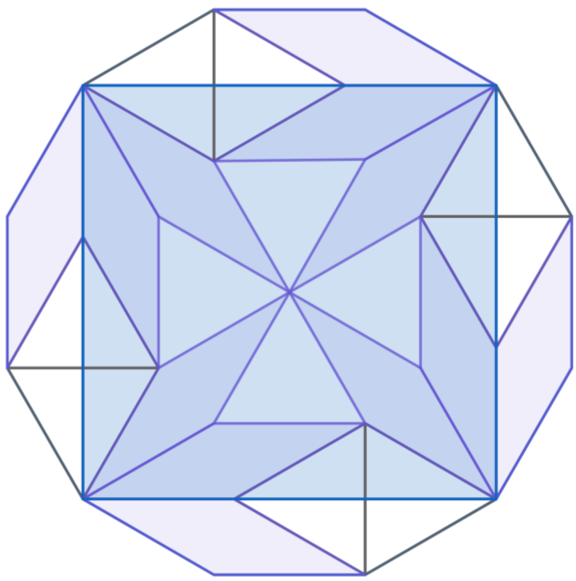


3.



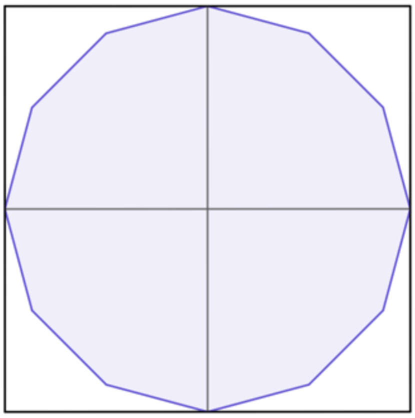
What fraction of the area of this regular dodecagon is occupied by the square?

We can dissect the dodecagon into 12 equilateral triangles and 6 squares, or into 12 equilateral triangles and 12 30° rhombuses (letter *c* in example **1.**). We'll do the latter.



Because rhombuses and triangles are present in equal numbers, we can take as our area unit a = (1 triangle + 1 rhombus). The required fraction is then $\frac{8a}{12a} = \frac{2}{3}$.

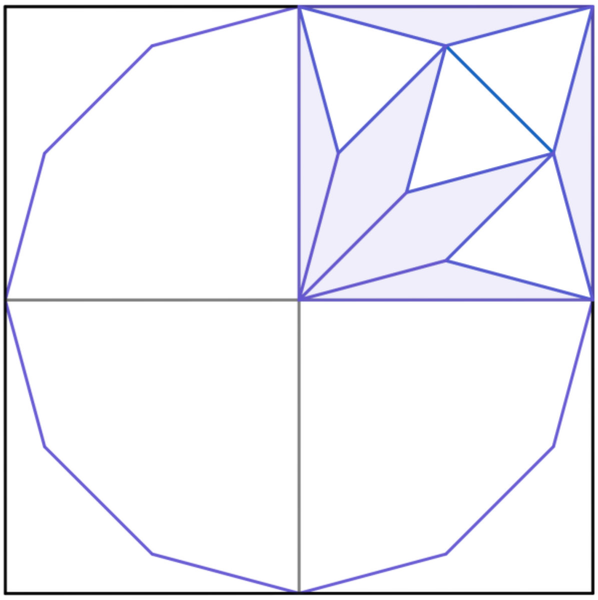
4.



What fraction of the area of this square is occupied by the regular dodecagon?

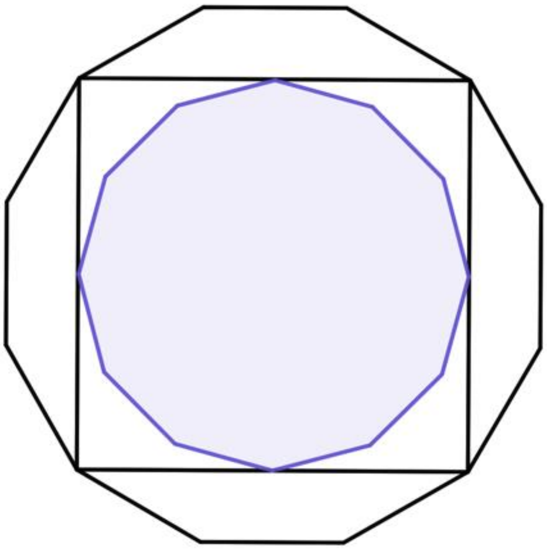
Several famous dissections solve this problem. We shall again use triangles and rhombuses. By symmetry we only need consider one of the four smaller squares. As in **3.** we can use our area unit *a*.

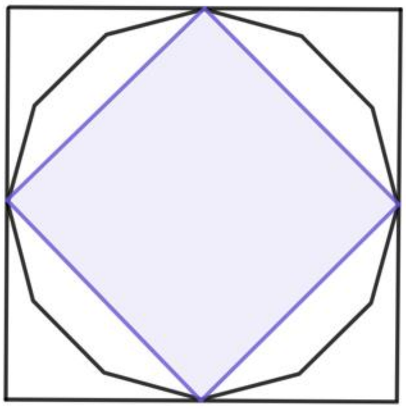
The required fraction this time is $^3/_4$.



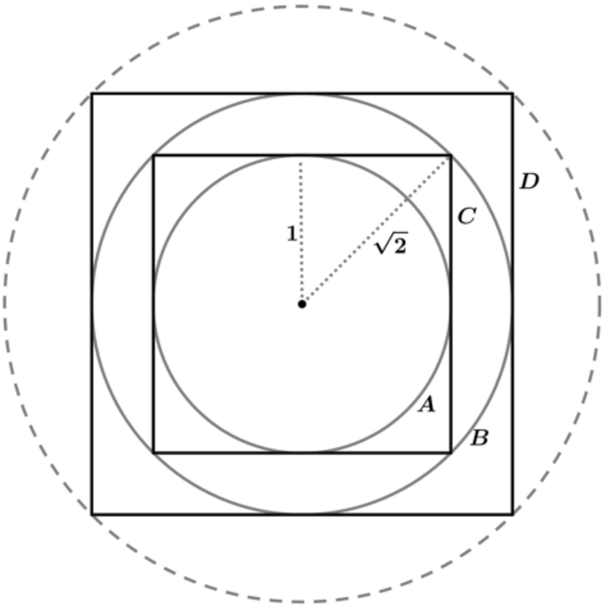
5. What fraction of the area of the outer dodecagon is occupied by the inner dodecagon below?

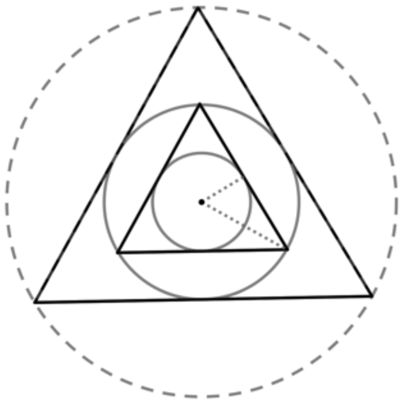
Now that we know the answers to **3.** and **4.**, all we have to do is multiply them. The answer is $\frac{3}{4} \times \frac{2}{3}$ but, by the commutative law, it's also $\frac{2}{3} \times \frac{3}{4}$. The geometric significance of this is that, instead of nesting a dodecagon-in-a-square-in-a-dodecagon, we can nest a square-in-a-dodecagon-in-a-square:



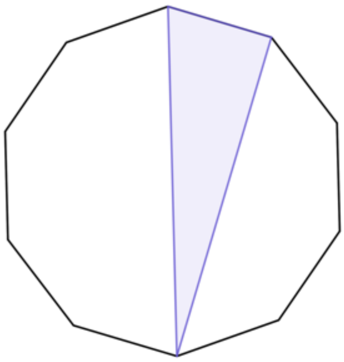


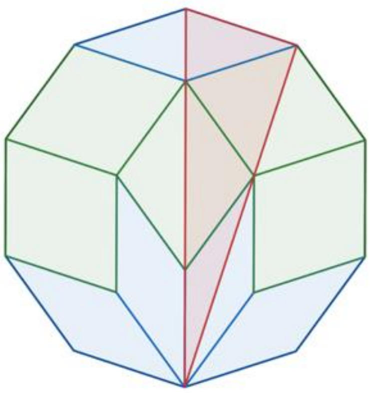
We see immediately that the answer is $^1/_2$. This tells us that the intermediate figure need not be a regular dodecagon, it could be an octagon or a 16-gon, or any regular polygon with 4*n* sides. As $n \to \infty$, the figure approaches a circle-in-a-square-in-a-circle. If the inner circle has radius 1, the outer has radius $\sqrt{2}$ so that the area ratio is $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$, as before. The nestings constitute a series. Circle *B* has twice the area of circle *A*; square *D* has twice the area of square *C*. If we change the regular polygon, this ratio will change. With the equilateral triangle, the area factor is not 2 but 4, as we see from the dotted '1-2-$\sqrt{3}$' right triangle:





6.





What fraction of the area of the regular decagon is occupied by the triangle shown, a region defined by a side and two adjacent diagonals?

Every regular polygon with 2k sides can be dissected into rhombuses. If k is odd, there are $\frac{k-1}{2}$ distinct shapes, k of each. If k is even, there are $\frac{k}{2}$ distinct shapes, $\frac{k}{2}$ squares and k of the other $\frac{k-2}{2}$ shapes.
We can make up composite units from these as we did for the rhombuses and triangles in the dodecagons.

With p and q for the two rhombus types here, we let $a = p + q$, and we have the fraction $\frac{a}{5a} = \frac{1}{5}$.

It turns out that, for the triangle defined as above, the fraction is always $\frac{1}{k}$.