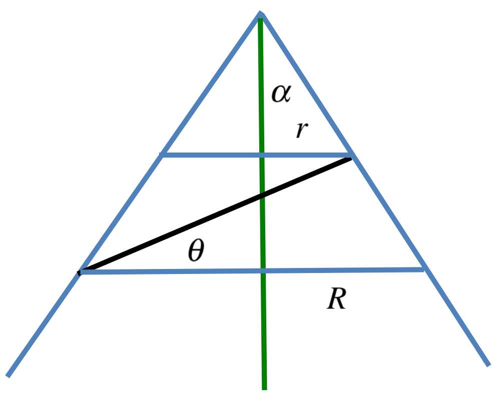
Find the condition for a right circular cone of semiapical angle α and a right circular cylinder of radius ρ to intersect in a plane section.

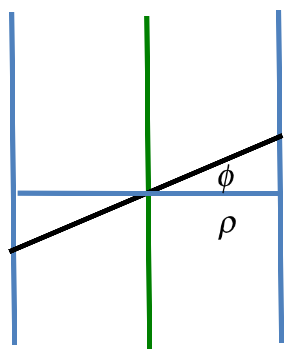
Any plane section of a cylinder which makes an angle > 0 with the axis is an ellipse. [**A**]
Any plane section of a cone which makes an angle > α with the axis is an ellipse. This condition implies that the plane cuts the axis. [**B**]

Thus the exercise reduces to achieving congruent ellipses, i.e. those with equal a and b, (the lengths of the semimajor and semiminor axes respectively) in the two cases.

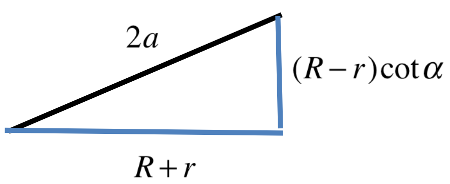
Choose any elliptical section of the cone, thus specifying *a* and *b*, and also the radius ρ of the cylinder, which must equal b. The ellipse and cone axes cannot be skew, for then a diameter of the cylinder would not correspond to the minor axis of the cone ellipse. Given that the cone and cylinder axes are coplanar, it remains only to determine the angle ψ the cylinder axis makes with the cone axis.

Here are cross-sections through the axes of the respective figures. In the case of the cone the ellipse is tangent to two circular sections, one of radius r, the other of radius R.





$$b = \frac{r+R}{2} = \rho$$
$$\Rightarrow R = 2\rho - r \quad \textbf{(I)}$$



Applying Pythagoras' Theorem in the above triangle and substituting from (**I**) gives

$$(2a)^2 = (2\rho)^2 + (2\rho - 2r)^2 \cot^2\alpha \quad \textbf{(II)}$$

$$2a = 2\rho\sec\phi \quad \textbf{(III)}$$

From (**II**) and (**III**) we have $\phi = \arctan\left[\left(\frac{\rho - r}{\rho}\right)\cot\alpha\right]$

But, using trigonometry in the above triangle and (**I**) again, we also have

$$\theta = \arctan\left[\left(\frac{\rho - r}{\rho}\right)\cot\alpha\right] = \phi$$

Since $\theta = \phi$, $\psi = 0$. Combining this condition with [**A**] and [**B**], we conclude that we require the axes of cone and cylinder to be parallel and the cone axis to fall within the cylinder.