Intervals between numbers coprime to a particular positive integer

Let N be a positive integer and consider the sequence of integers coprime to N.

EXAMPLE $(N = 30)$

$$\dots, -7, -1, 1, 7, 11, 13, 17, 19, 23, 29, 31, 37, \dots$$

Such a sequence contains a block of $\varphi(N)$ integers, from 1 to $N - 1$. Successive blocks of $\varphi(N)$ integers can then be formed by adding $\pm N, \pm 2N, \pm 3N, \dots$ to each integer of the first block.

The intervals between successive integers of the sequence therefore form a repeated pattern of $\varphi(N)$ differences, with at least one difference of 2 (the difference between $\gamma N + 1$ and $\gamma N - 1$). Note that the sum of $\varphi(N)$ successive intervals will add to N (the difference between $N + 1$ and 1).

EXAMPLE $(N = 30)$

$\varphi(30) = 1\text{x}2\text{x}4 = 8$. The repeated sequence of 8 intervals is:-

$$6, 4, 2, 4, 2, 4, 6, 2.$$

Then $6 + 4 + 2 + 4 + 2 + 4 + 6 + 2 = 30$.

Counting strip sequences

For $1 \le i \le n$, let p_i be distinct primes and let a_i be positive integers. Define $N = \prod p_i$.

By the Chinese Remainder Theorem, there is an integer X such that

$$X + i \equiv 1 \ (\text{mod}\ {p_i}^{a_i+1}), \text{ for } 1 \le i \le n.$$

For the consecutive numbers

$$L, L+1, \dots, L+n-1$$

to satisfy the required 'counting strip property' w.r.t. the powers ${p_i}^{a_i}$, we require

$$L + i \equiv 1 \ (\text{mod}\ {p_i}^{a_i}),\ L + i \not\equiv 1 \ (\text{mod}\ {p_i}^{a_i+1})$$

$$\Leftrightarrow L = X + \gamma {p_1}^{a_1} \dots {p_n}^{a_n}, \text{ with } \gamma \text{ coprime to } N.$$

The required results therefore follow immediately from the results for numbers coprime to N.

EXAMPLE $\underline{\text{(Factors 3, ,4, 5)}}$

A suitable value for X is 423. Then

$$X \equiv 0(\text{mod } 3^2), X+1 \equiv 0(\text{mod } 2^3), X+2 \equiv 0(\text{mod } 5^2).$$

So $L = 423 + \gamma 60$, with γ coprime to 30. Then

$\gamma = -7$ gives the sequence (3,4,5),

$\gamma = -1$ gives the sequence (363,364,365),

etc.