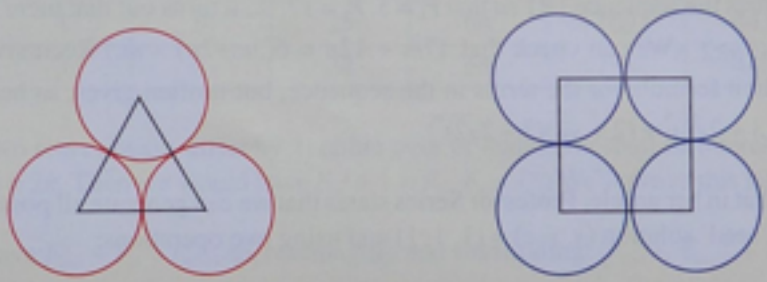
BOXLESS PACKING
Why Mathematics is Necesssary

Small challenges, bigger challenges, a very big challenge

Visitors to *The Magic Mathworks Travelling Circus* are faced with practical puzzles they can only solve by means of mathematics. I'm going to give you a clue to solving one of them right at the start of this piece. But if you think this puts you one up on the innocent visitor, think again! You may be quick to solve that particular puzzle, but the puzzle is only there to pose a further one. To solve *that* you will need to use mathematics. And success *there* will only lead to a further problem, one posed to the mathematician/astronomer Johannes Kepler in 1611 and only solved by the mathematician Thomas Hales in 1998. I'm not expecting you to crack that one now – but in, say, 10 years' time. The thing is, Kepler made a conjecture which he was too good a mathematician to know he could not *prove*. What Hales did was offer a proof good enough for the mathematical community to accept. But why shouldn't you come up with a better one?

Circle packings in small boxes, large boxes, no boxes

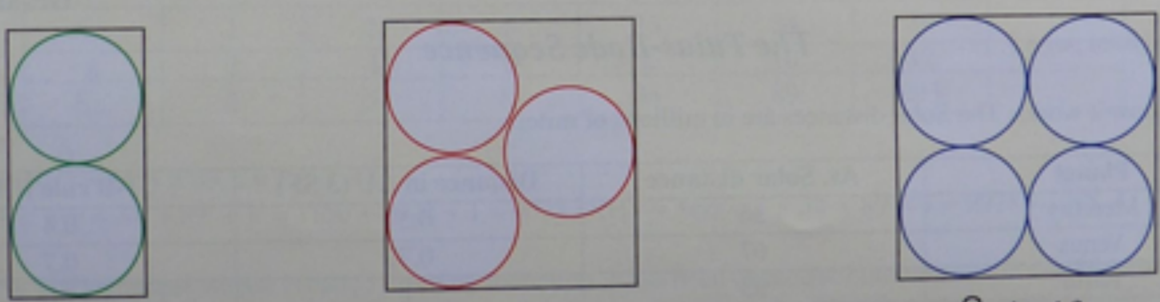
I offer you two ways to pack circles and ask, "Which is better?". The object of this piece will be to refine our use of the word 'better'.



T S

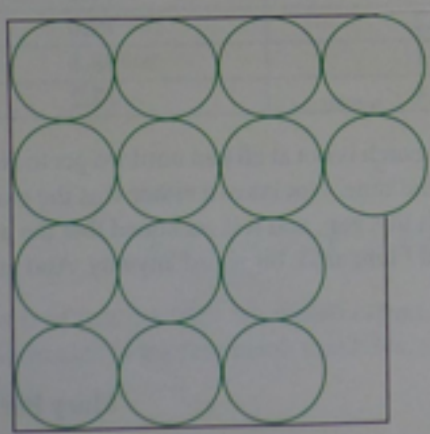
In the T arrangement the centres of circles round a space form a triangle; in the S arrangement, a square.

We shall fit circles into rectangles of increasing size – and carefully chosen shape – and see which arrangement enables us to include more.



Neither T wins 3:2 S wins 4:3

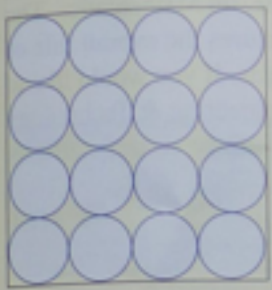
The next rectangle is the one we use in the *Circus*. The visitor must fit 14 'people'(discs) in a 'lift' (shallow tray). Their first, failed, attempt usually looks like this:



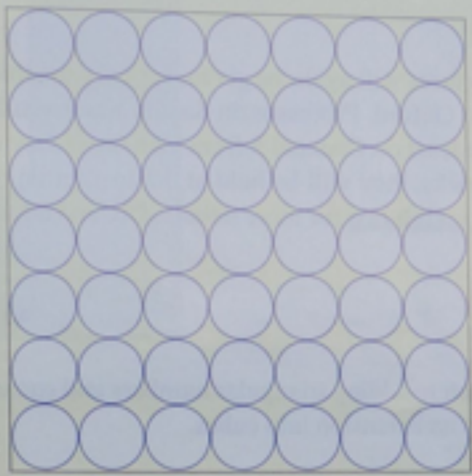
Alongside, however, is a piece of apparatus with which they can perform an *analogue* experiment. (We met one of these before in SYMmetry*plus* when we used soap films to show the shortest total length of motorway need to connect *n* cities.) Hundreds of tiny wooden balls are confined to the floor of a tray by a Perspex sheet. When you tilt the tray, they roll to one edge and take up the pattern with the least potential energy, which turns out to be the T arrangement. A little window is drawn on the Perspex so that the visitor can see the very solution he or she is seeking.

So T wins 14:12.

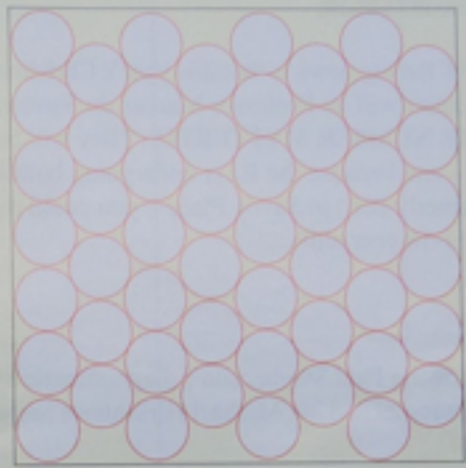
But let's continue to enlarge our rectangles



S wins 16:14



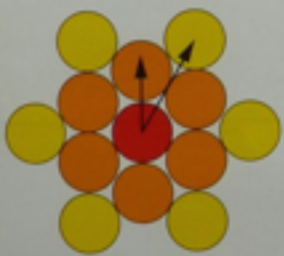
S wins 49:46



T wins 52:49

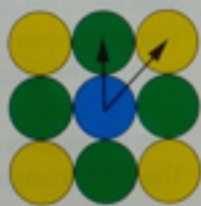
As long as it our intention to fill a box – however large – our question, "Which packing is better?" will only have an answer for that particular box. We can only answer the question for an infinite box – in other words, the plane – if we have a measure for closeness of packing.

Before we define that, I want to compare the geometry of the two arrangements.



T

There are 6 neighbours in immediate contact with a chosen circle
– but the next 6 are quite a long way out.



S

There are only 4 neighbours in immediate contact with a chosen circle
– but the next 4 are quite close.

It is not obvious that, were we to go out far enough, one arrangement would win over the other.

Our measure of closeness of packing is a simple one: how much space each circle occupies.

T

S

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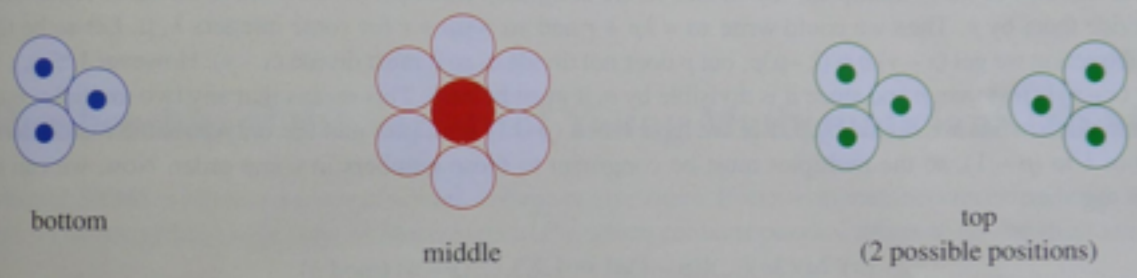
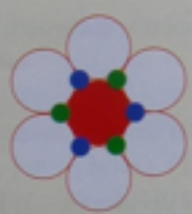
Boxless Packing

In the S arrangement we have a square (4 sides); in the T arrangement, a regular hexagon (6 sides). As the number of polygon sides increases you can see that we get closer and closer to the circle being contained. It's therefore clear without doing any calculation that the area of the hexagon is smaller than that of the square. On our measure, therefore, T is 'better'. (By the way, you can do this calculation and work out the dimensions of all the figures in this article using no more than Pythagoras' Theorem.)

Now we come to 'Kepler's Packing Problem'. We move into 3 dimensions. The circles are now spheres. Imagine 3 layers: blue at the bottom, red in the middle, green on top:



We assemble our model. (There are, as you see, alternative arrangements, but that's not our concern here.)



We find the marked (solidly coloured) sphere touches 3 below, 6 in its own layer, 3 above: 12 in all. But what if the surrounding spheres didn't quite touch – or didn't all quite touch – and we somehow squeezed in a 13th, would that lead to a closer packing overall? Hales addressed himself to this question and found that no alternative arrangement was better than the simple one. No one doubted that he would find this. But the moral of the story – and the reason for my subheading – is that years of careful work were needed to establish the 'obvious'. However, one attraction of mathematics is that it's always possible that someone will establish a result by a shorter proof.

And that someone may be you.

Paul Stephenson
The Magic Mathworks Travelling Circus

If you're feeling brave you might like to try to work out just how much space is occupied by the spheres when packed like this – and as someone said, greengrocers have always known this was the best method!

Ed