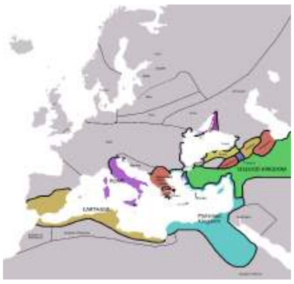
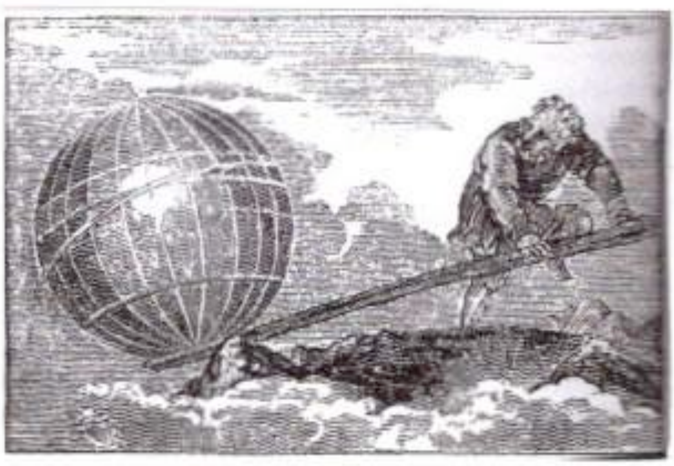
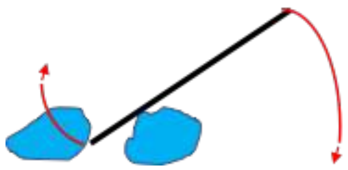
Archimedes' best ideas

Nearly two millennia later Galileo was to say that the book of nature was written in the language of mathematics. But Archimedes was arguably the first person to read this language. In other words, he was the first appied mathematician, the first mathematical modeller. The point of this workshop is to show what it means to view the world through the eyes of a mathematician.

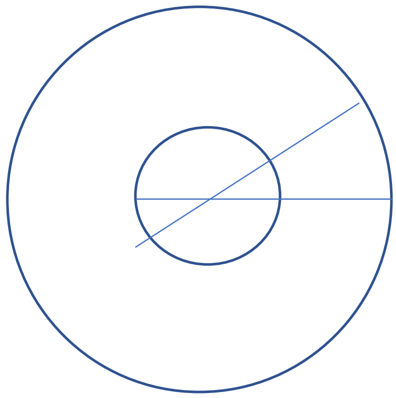
We shall use modern terms, particularly 'force'. For this, Archimedes used a word we would now translate as 'magnitude', indicating a scalar not a vector quantity, but the concept is implicit in his work.

materials needed	*activity, information to give the children (A. = Archimedes)*
	*We're in the 3rd century B.C. Here is a map of the Mediterranean.* Locate Syracuse, indicate reach of Roman, Carthaginian, Greek empires. *When Syracuse sided with Carthage against Rome, it meant war – of which more anon. A. was scientific adviser to the king. He studied in Alexandria. This was like moving from Cambridge, England to Cambridge, Massachusetts. The Alexandrians didn't just pursue academic subjects but also engineering – again, more about this anon.*





For hundreds of thousands of years we – not to mention intelligent animals like the Caledonian crow - have been using levers with an intuitive understanding. The Greeks before A. understood the geometry – it's just enlargement or similar figures – because that's what you see. But A. was arguably the first to see (in quotes) what was invisible, the forces involved, thus founding the science of statics, the study of forces in equilibrium.



Mathematical balance

E1 *Teacher demonstration*

Show different arrangements of the masses. Request the rule, 'the law of the lever'.

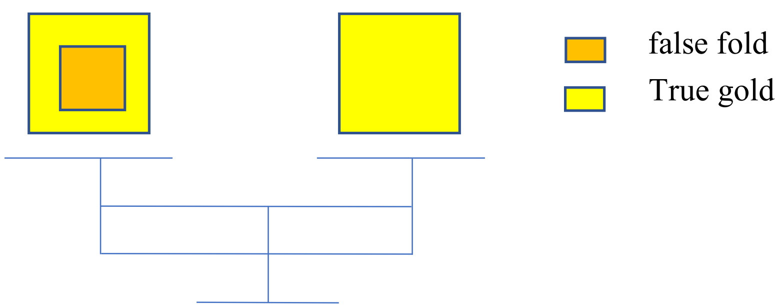
[L.H. force x L.H. distance = R.H. force x R.H. distance.]

Set the scene:

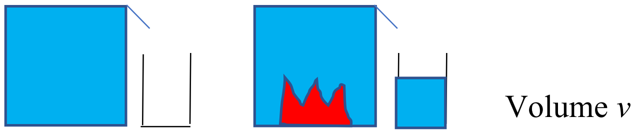
King: "Archimedes, the court goldsmith has wrought me this crown but it seems to me somewhat light and I suspect he has admixed some false gold."

Lead the children through A.'s imaginary thought sequence:

1. I am concerned here with *density*, mass per unit volume.
2. If the man is guilty I shall find that the density of the crown material is less than that of gold.
3. If I had identical volumes of gold and of the crown material, and I were sure my scales were reliable, I could settle the matter:



4. But how can I find the volume of an irregular shape? EUREKA!



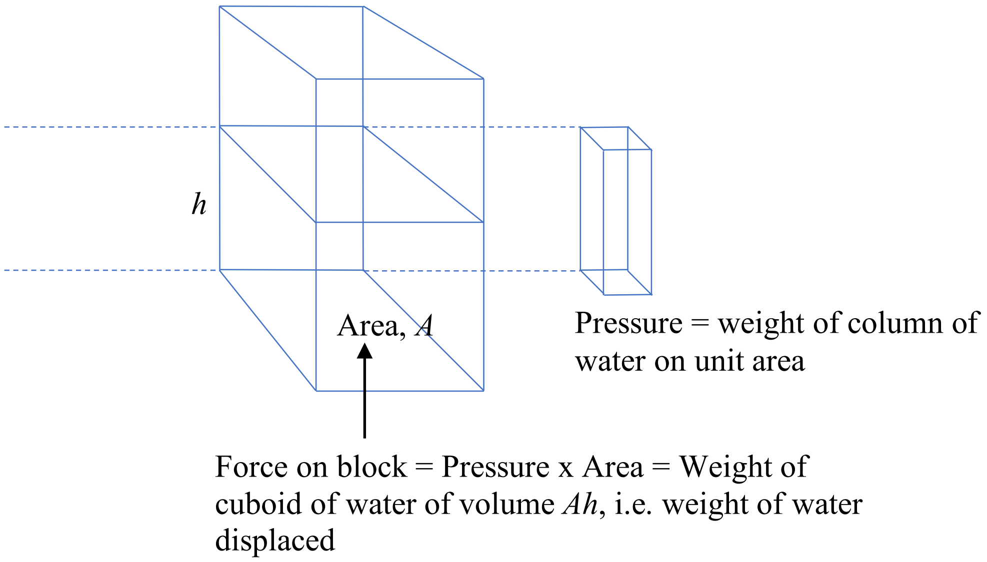
This is the *displacement principle.*

So I could cut a volume *v* from a block of gold and weigh it against the crown on my scales. But I worry that, if the difference is slight, my untrustworthy scales might send an innocent man to his death.

5. I have a better idea:

There is a net upward force on a body due to the volume of fluid displaced. This is the *weight* of the displaced volume. And this is true however deeply submerged the object.

This is *Archimedes' principle*.



6. If this is true, a denser object will displace a smaller fraction of its weight than a lighter one.

7. I can compare the two fractions with my lever.

8. If I test a sample of false gold too, I can work out what fraction of the crown by volume is false gold.

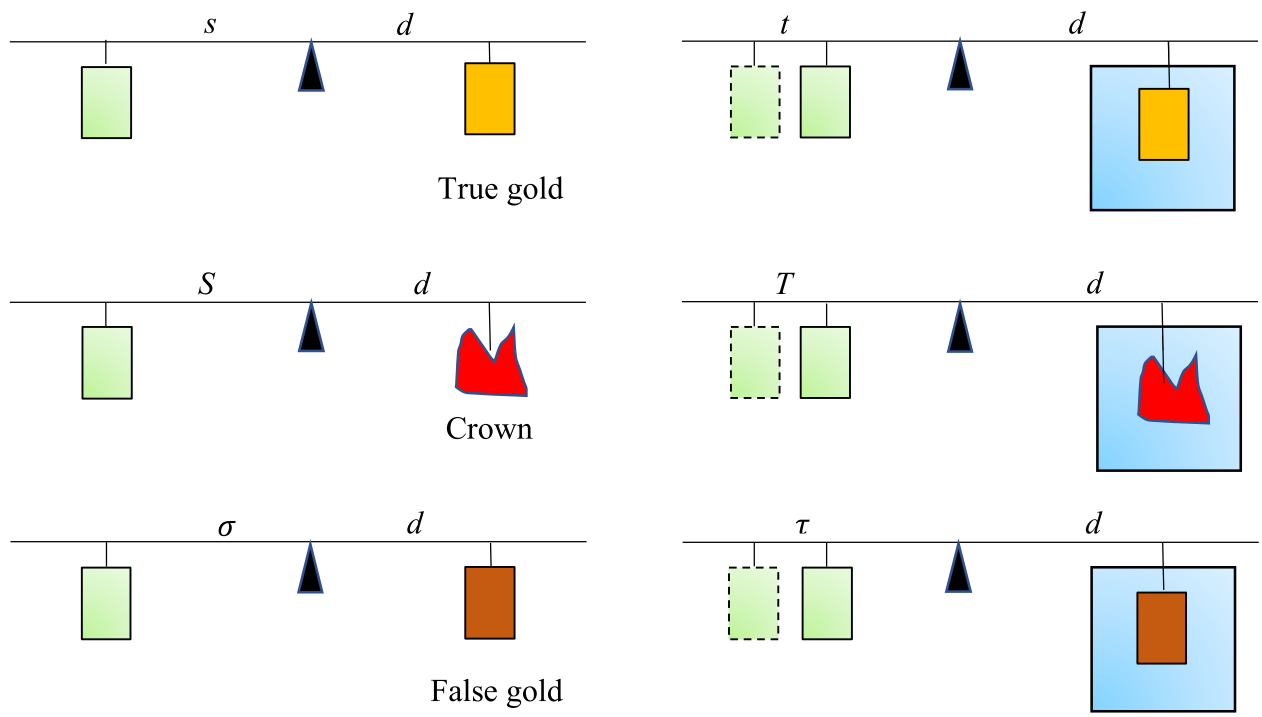
Ff. x 15:

E2 *Class experiment*

Meccano balance,
Small washer as counterpoise,
Big washer (gold),
Clay lump (false gold),
Bolt + clay (crown),
Cup,
Water jug

Distribute these worksheets:

Choose a d value, (which remains fixed), make these weighings, put your readings in the boxes, and work out the densities.



$s =$ ☐ $t =$ ☐ The density of true gold relative to water, $\rho_G = \frac{s}{t} =$

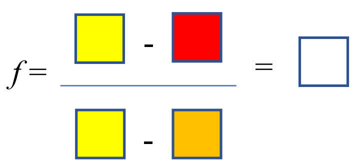
$S =$ ☐ $T =$ ☐ The density of the crown material relative to water, $\rho_C = \frac{S}{T} =$

$\sigma =$ ☐ $\tau =$ ☐ The density of false gold relative to water, $\rho_F = \frac{\sigma}{\tau} =$

ρ_C is a weighted mean of ρ_G and ρ_F. Let the fraction by volume of the false gold in the crown be f.

$$\rho_C = \frac{f\rho_F + \rho_G}{f+1} \Leftrightarrow f = \frac{\rho_G - \rho_C}{\rho_G - \rho_F}.$$

So now use your density values to work out f.



Card, Scissors, Pin, Plumbline, Pen, Straight edge

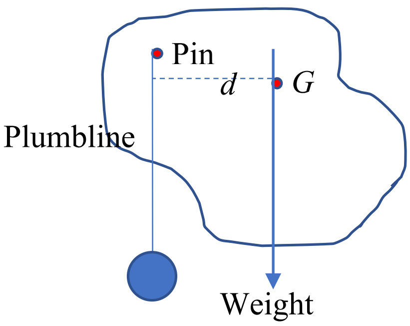
E3 *Class experiment*

The law of the lever, or principle of turning moments, has an important application which we need before turning to A.'s work on flotation.

When the force of gravity acts on a body, it acts on every part of it but, when we take the object as a whole, we can identify one point in it through which the force acts: its centre of mass, G in the figure below. Real objects are 3-dimensional but we shall work in just one plane.

Cut the card to a shape of your choice, e.g. Archimedes' profile. You don't know where *G* is but insert the pin somewhere, attach the plumbline and let both hang.

There is now an unbalanced turning moment, *Weight* x *d*, which swings the lamina down until *G* lies vertically under the pin.



Ask a colleague to make a mark on this vertical line so that you can rule it later.

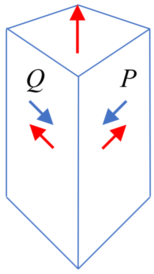
Choose a new point of suspension and repeat the experiment. Rule the two lines. Their point of intersection is *G*.

When a ship floats, the force on the body of displaced water, which we used in the 'King's crown' experiment, acts up through its centre of mass. As the ship heels over, this body of water changes its shape and the centre of mass moves accordingly. We distinguish:

(a) *the centre of mass of the ship, G, which is a fixed reference point within it,*

(b) *the centre of mass of the body of displaced water corresponding to a particular heel angle of the ship. This is the centre of buoyancy, H.*

We are going to have our ship heel over at 45° and calculate whether it will right itself or capsize. Because ships, and the bodies of water displaced by them, have complicated shapes, our ship will be a square prism. Because the block is a cuboid, we know that the position of G in a face holds for all *planes parallel to it. By the symmetry of the square, if the net turning moment pulls the block upright in plane P, it will do the same in plane Q. But if the block is pulled over in plane P, it will also be pulled over in plane Q so that, by symmetry, it will tip over in a diagonal plane and float with an edge upright.*



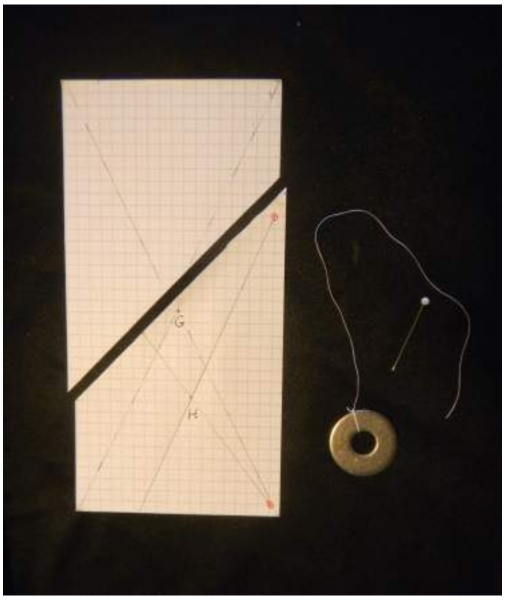
*The density of our wood relative to water is 8/15. This means that the floating block would sit upright with 8/15 of its height under water.(**A**) shows the result. (We have scaled the model to make the arithmetic easier.) Our 45° angle also makes it easy to work out the dimensions in (**E**), which is just (**B**) set upright. (**B**) shows the position of the block when we release it.*

We must find the exact positions of G and H. If H is to the left of G, the net turning moment will be clockwise and will capsize the ship; if H is to the right of G, the net turning moment will be anticlockwise and will right it.

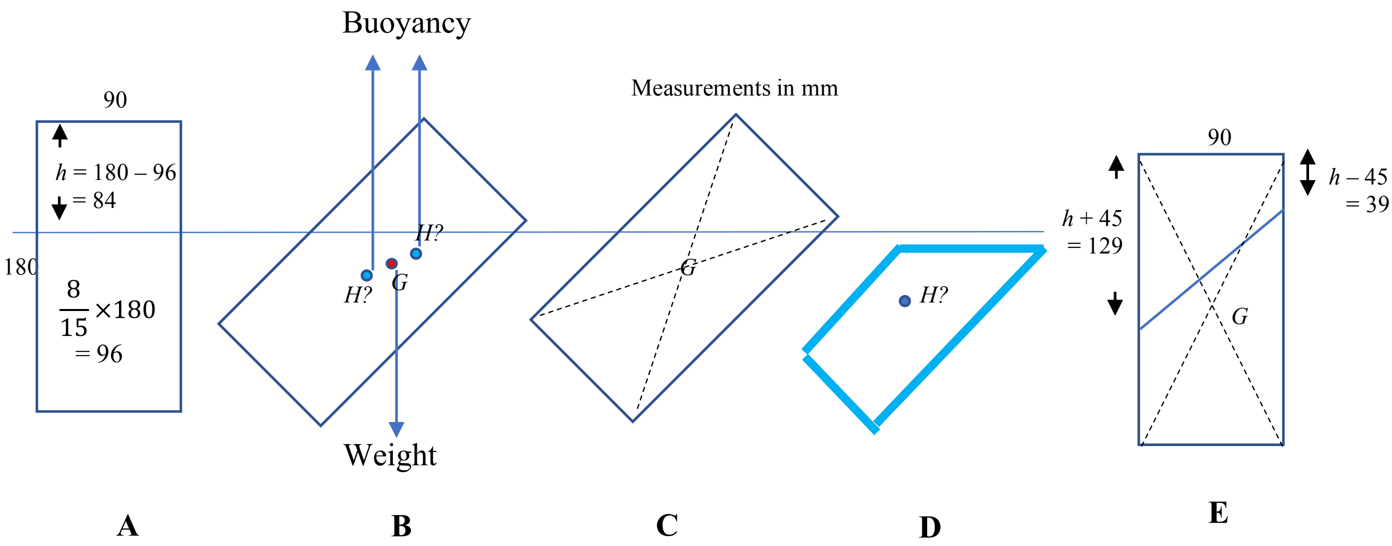
*The position of G we know by symmetry (**C**). H we shall find by making a cardboard copy of the blue piece (**D**) and finding its centre of mass by the method of the last experiment.*

Ff. x 15:

Square prism, with cross-section 2:1,
Cardboard rectangle corresponding,
Straight edge,
Scissors,
Pin,
Plumbline,
Pen,
Water tub



E4 *Class experiment*



Mark the points on the card shown in (**E**).
Rule the waterline.

Construct the position of G (**C**, **E**).

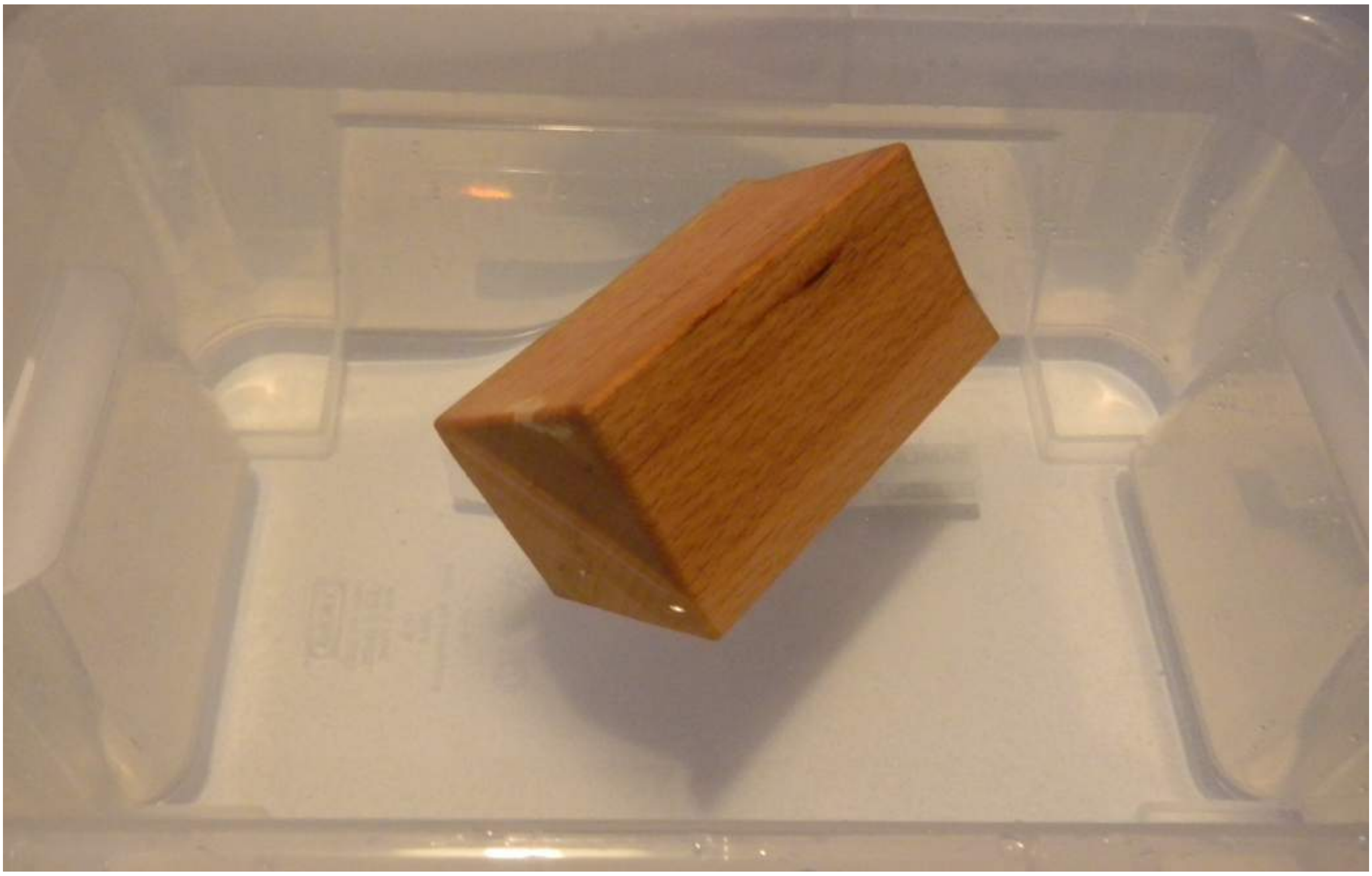
Cut along the waterline.

Perform the previous experiment on the lower part to find H.

Is H below G (therefore to the left on the block)
or above G (therefore to the right on the block)?

What, therefore, are you expecting to happen to the block if released in the 45° position?

Make the experiment.



With this work A. established the science of <u>*hydro*</u>*statics, which is today as A. left it. He was also the world's first naval architect: he designed the biggest ship of its time, The Syracusa, a present from the king of Syracuse to the ruler of Alexandria.*

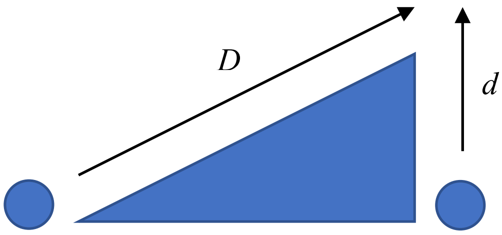
As a means to raise heavy loads, the inclined plane has been in use for thousands of years.

Ff. x 15:

Wedge,
Rollers (2)

E5 *Class experiment*

Begin with one roller at the foot of the slope, the other on the sheer side.
Use one hand to raise the second roller in a straight lift, the other to roll the first up the slope.



On one side you exerted a small force through a big distance; on the other, a big force through a small distance. If the slope was frictionless, we would have fD = Fd. Let's call this 'the machine equation' because we're going to meet more examples. (Nowadays we would equate the expressions to the work done and the potential energy acquired, but Archimedes did not express himself in this way.)

Though he may have been anticipated by a heavy bronze device which watered Sennacherib's gardens at Nineveh, and perhaps by an earlier Greek, Archytas, Archimedes is credited with the invention of the screw used to lift water from a lower level to a higher. He applied it to the problem of emptying water from the holds of ships. The screw – whether in that device or as a means of attaching one object to another – is the inclined plane coiled up.

Acetate triangle

E6 *Teacher demonstration*

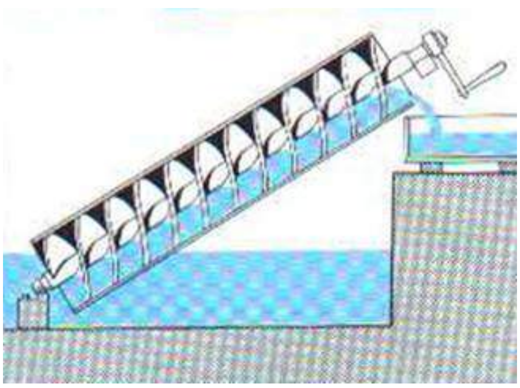
Coil the triangle to show that the screw is just the inclined plane rolled up.

Screw clamp,
Block

E7

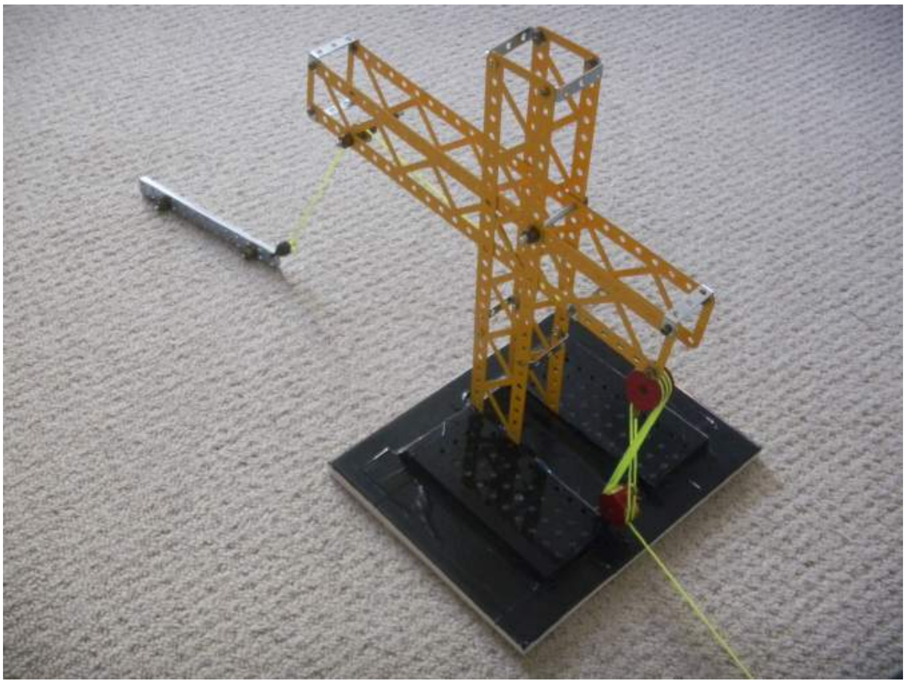
Tighten the clamp on the block and ask the children for an explanation of its action.

Introduce the screw as a lifting device.



Archimedean screw **E8** *Demonstration by 2 pupils*

As anticipated, in Archimedes' time Rome declared war on Syracuse and laid siege to the port. It was here that all Archimedes had learned of engineering in Alexandria, and all the improvements he had made to existing devices, came into play. He constructed huge catapults for bombarding the ships and, most famously, the Claw, for hoisting the ships out of the water so that all the crew fell out. Here is a possible version:



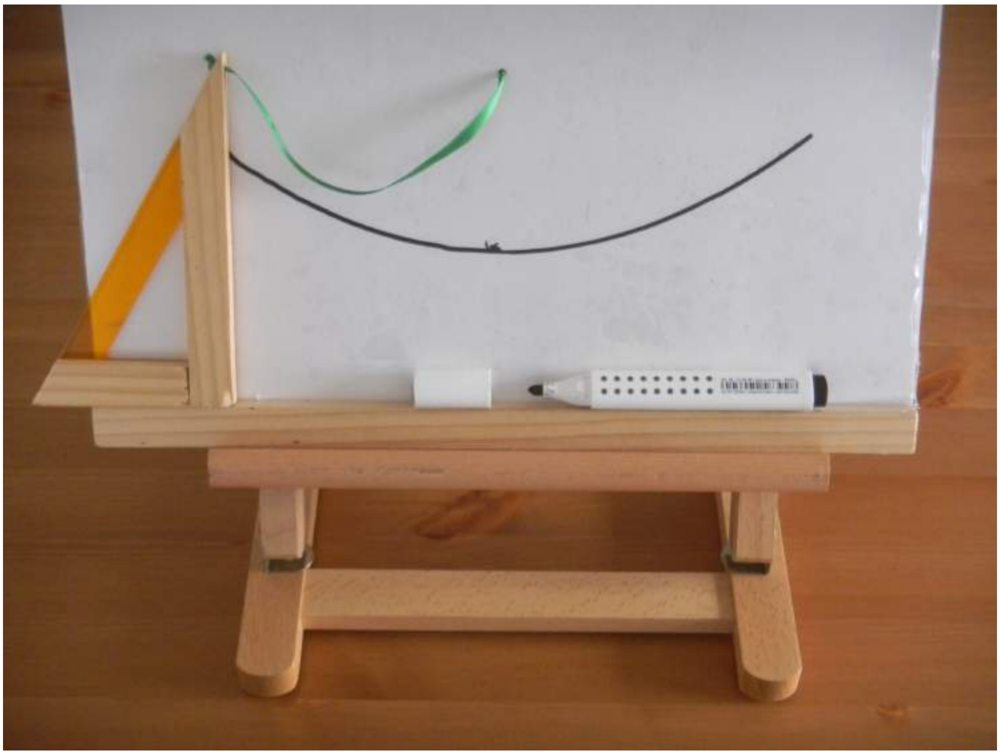
The Claw apparatus **E9** *Demonstration by 2 pupils*

Draw attention to the pulley blocks. Ask how they work.

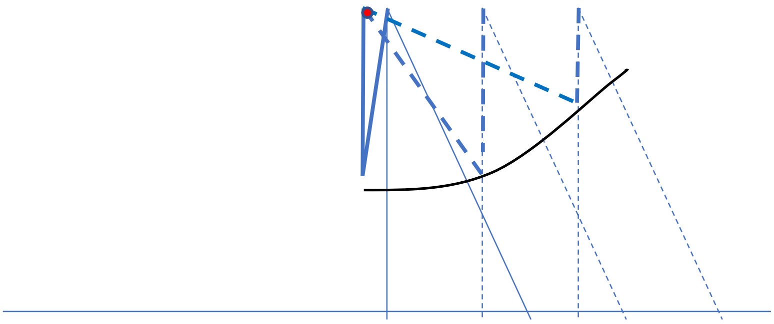
A. impressed the king by showing how, with the help of pulley blocks, he could move a fully laden ship with one hand. (According to one tradition this was the Syracusa, which had got stuck on the slipway.)

That Archimedes used parabolic mirrors to ignite the Roman ships was probably just propaganda put out to alarm a navy already terrified by 'the Claw', But Archimedes was certainly familiar with the focusing property of the parabola.

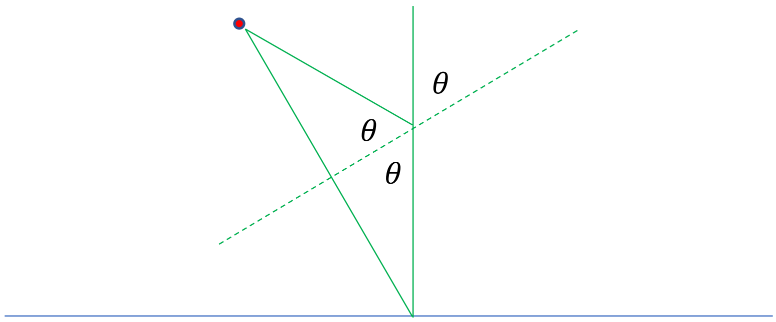
Parabola apparatus A **E10** *Demonstration by teacher + 2 pupils*



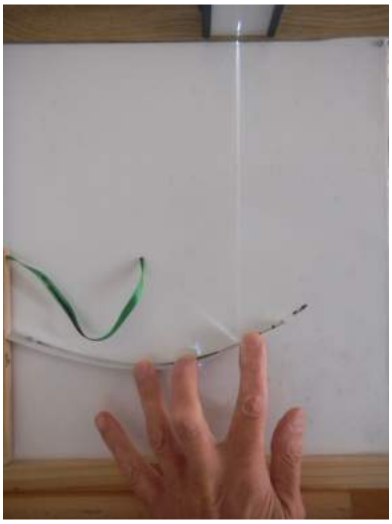
Insert the black dry-wipe pen in the string loop, hold it against the set square and slide the set square away from the nail, thus producing half a parabola by virtue of the fact that the distance from the pen to the nail (focus) is equal to that from the pen to the rim (directrix). Turn the set square over and repeat to produce the symmetrical half.



Return the apparatus to some intermediate position. Use the straight edge to draw a line in green from the right angle to the nail. Use the mirror to construct the perpendicular bisector. Also trace in green the front edge of the set square and the line from the string to the nail. Label the diagram which results to show that the front edge of the set square and the line of the string make equal angles with the perpendicular bisector, which is tangent to the parabola. By the law of reflection a ray parallel to the axis of the parabola therefore passes through the focus:



Parabola apparatus A + B:



1st pupil:

Flex the mirror strip so that it follows the line of the parabola.

2nd pupil:

Run the light box with its collimated beam along the top of the apparatus. The reflected beam will be seen to pass through the focus, lighting up the head of a glass-headed pin.

The surface of revolution of a parabola is a paraboloid. The focusing property leads to its use in reflecting telescopes and satellite dishes.

E11 *Teacher demonstration*

*Card circle sector (**A**),*
*Card annulus sector (**B**),*
Card hemisphere approximations, large and small,
Perspex hemisphere

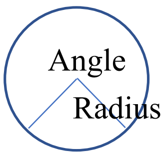
Here in white card are two geometrically similar approximations to a hemisphere. The band of latitude $60° - 90°$ is formed by a cone; the bands $30° - 60°$ and $0° - 30°$ by frusta of cones.

Fold **A** to see that it makes a cone.

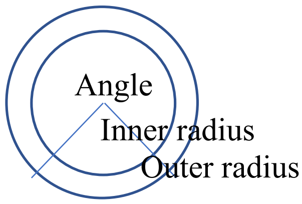
Fold **B** to see that it makes a cone frustum.

From the dimensions shown below, using simple proportion and the formula for the area of a circle, we could calculate the area of each, and therefore the surface areas of our approximate hemispheres.

A



B



Fit the acetate hemisphere over the smaller model.

Fit the bigger model over the acetate hemisphere.

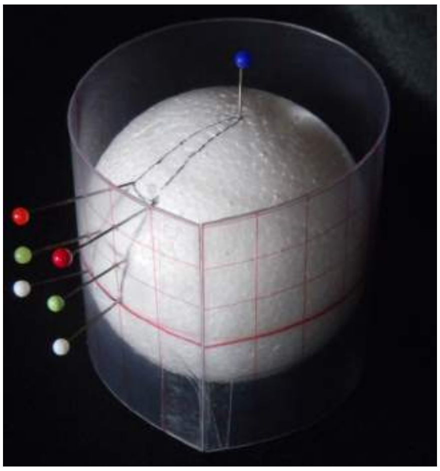
The true hemisphere is thus sandwiched between two approximations, one a bit too big, one a bit too small.

As a thought experiment, increase the number of latitude bands in each model, from 3 to 4 to 5 …

Thus Archimedes approached the true shape, and therefore the true area of the hemisphere, from above and below.

Ff. x 15: **E12** *Class experiment*

Polystyrene sphere,
Graduated cylindrical sleeve,
Glass-headed pins (4),
Pen



Fit a sphere in a sleeve.

Insert pins through the holes so that they enter the sphere horizontally along a radius of the circular section.

Remove.

Join up the pinpricks.

The curved region on the sphere has the same area as the cylindrical region on the sleeve.

E13 *Class experiment*

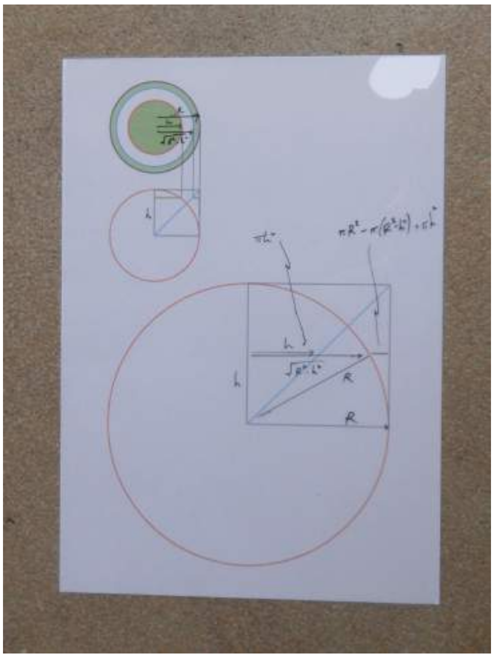
Cylinder-hemisphere-cone model

Study the model.
There are 3 parts, all of unit height and radius:

- A cylinder,
- A hemisphere,
- A cone.

Laminated sheet,
Dry-wipe pen

Now look at the diagram, which you can write on. Top left is a plan view of the model; under it, a vertical section, enlarged bottom right.



By using:

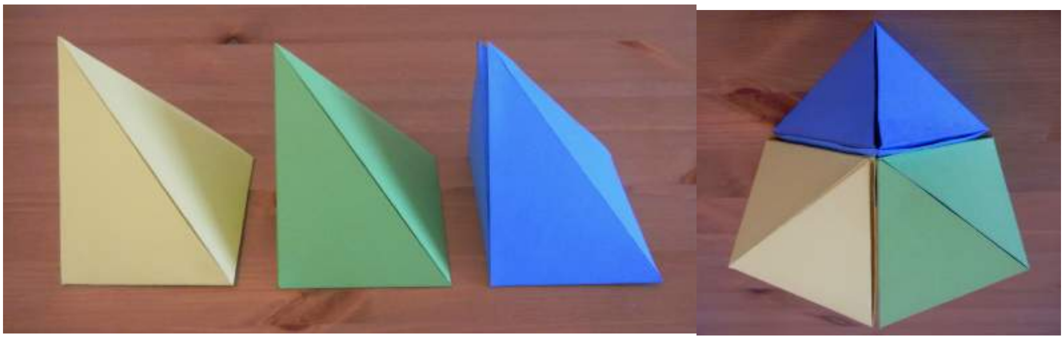
- The formula for the area A of a circle, radius ρ: $A = \pi\rho^2$,
- The formula for the area B of an annulus,

inner radius r, outer radius R:
$B = \pi(R^2 - r^2)$,

- Pythagoras' theorem,
- The fact that the blue line slopes at 45°

show that the areas of the green annulus and circle are equal.

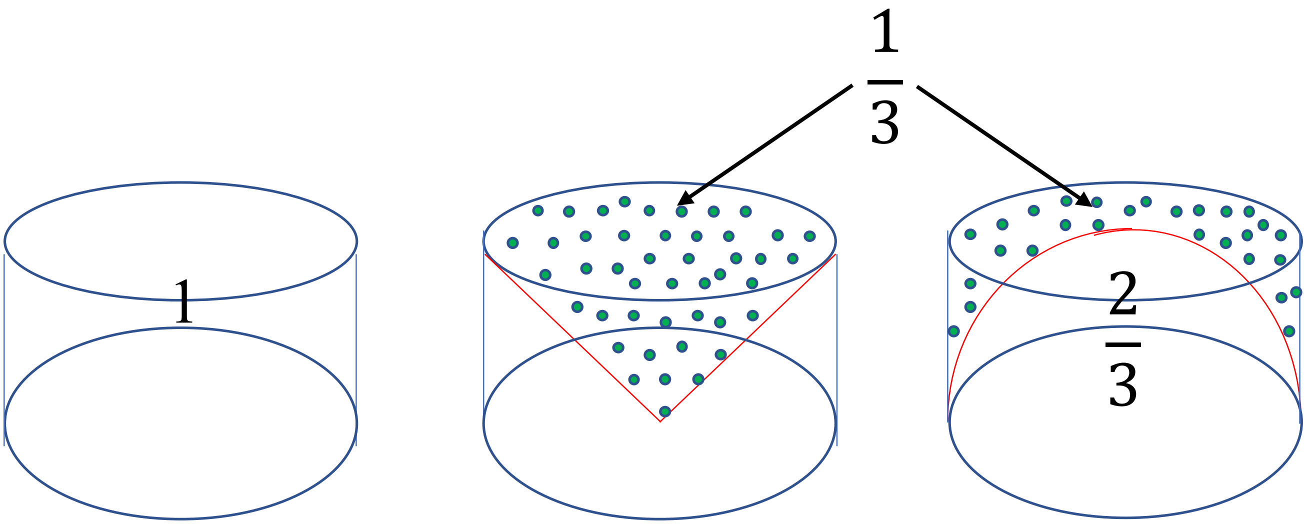
We now imagine those areas as the areas of thin slices stacked up to make solids. (*With this work A. launched the integral calculus, which you'll study later.*) The stacked disks make a cone. The stacked annuli make the gap between the hemisphere and the cylinder. A. knew that the volume of a cone (like a pyramid) was one third the base area times the height. We can show this by dissection in the case of 3 equal pyramids which make a cube:



Above dissection in card **E14** *Teacher demonstration*

Exhibit.

So he deduced the volume of the hemisphere by subtraction:



A. found this solution not as we've done but by thinking about the shapes hanging from a balance and using 'the law of the lever'. We only know this because of a copy of a work found last century concealed by another written over the top. (At the time this practice was common because the medium it was written on, called vellum, was very expensive.)

We're going to confirm in a rough physical way the result we've just found theoretically.

E15 *Demonstration by 2 pupils*

Polydron 'Sphera' pieces:
Quarter-hemisphere, -cylinder, -cone, -disk (4),
Board, Sand, Trowel, Spatula

Make a Polydron hemisphere.
Make a pile of wet sand.
Push the Polydron hemisphere down on the sand, swivelling it to and fro to shape a sand hemisphere.

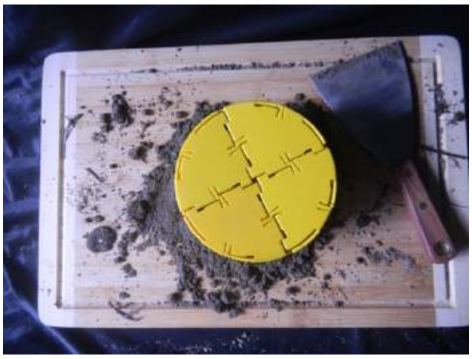
Make a Polydron cylinder.
Drop the cylinder over the sand hemisphere.

Make a Polydron cone.
Pack it with sand.

Trowel the sand from the cone into the cylinder and pack down.

.

Make a Polydron disk.
Fit the disk over the top.



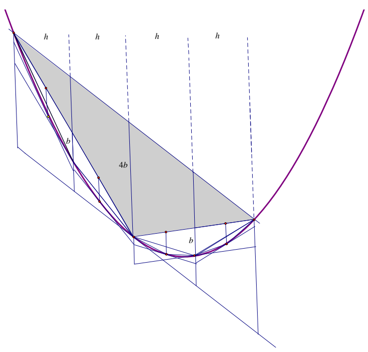
E16 *Class experiment*

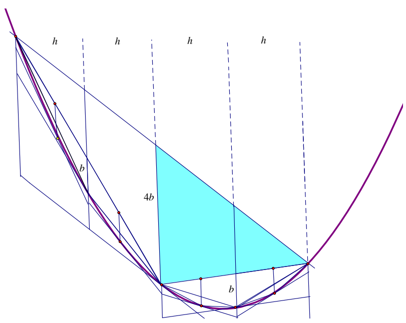
Work through this sequence.

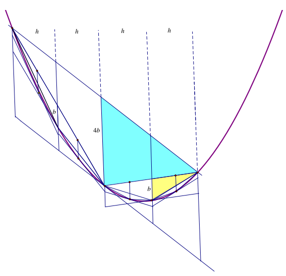
Apparatus

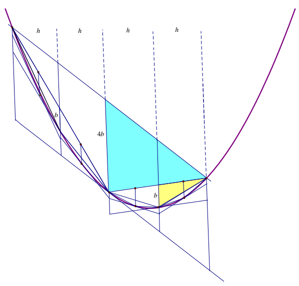
Figure with grey, **(A)**
Figure with blue, **(B)**
Figure with blue + yellow, **(C)**
Figure with blue + yellow + green **(D)**

Archimedes wishes to find the area between the chord and the parabola in terms of the grey triangle **(A)**, whose apex lies at the point of contact of the tangent parallel to the chord.
He builds up this area by packing smaller and smaller triangles in the remaining spaces. He knew two properties of the parabola:
1) the tangent point lies half way across the chord, measured perpendicular to the axis.
2) the base of the new triangle, measured parallel to the axis, (b) is ¼ of the old one ($4b$).









Let us see how the area grows. <u>Total units</u>

Sheet (**B**): 1 grey triangle = 2 blue triangles = 2 (half base times height)

$= 2\left(\frac{1}{2}.4b.2h\right) = 8bh = 1\,unit.$ 1

Sheet (**C**): 4 yellow triangles = 4 (half base times height)

$= 4\left(\frac{1}{2}.b.h\right) = 2bh = \frac{1}{4}\,unit.$ $1 + \frac{1}{4}$

Sheet (**D**) 8 green triangles = 8 (half base times height)

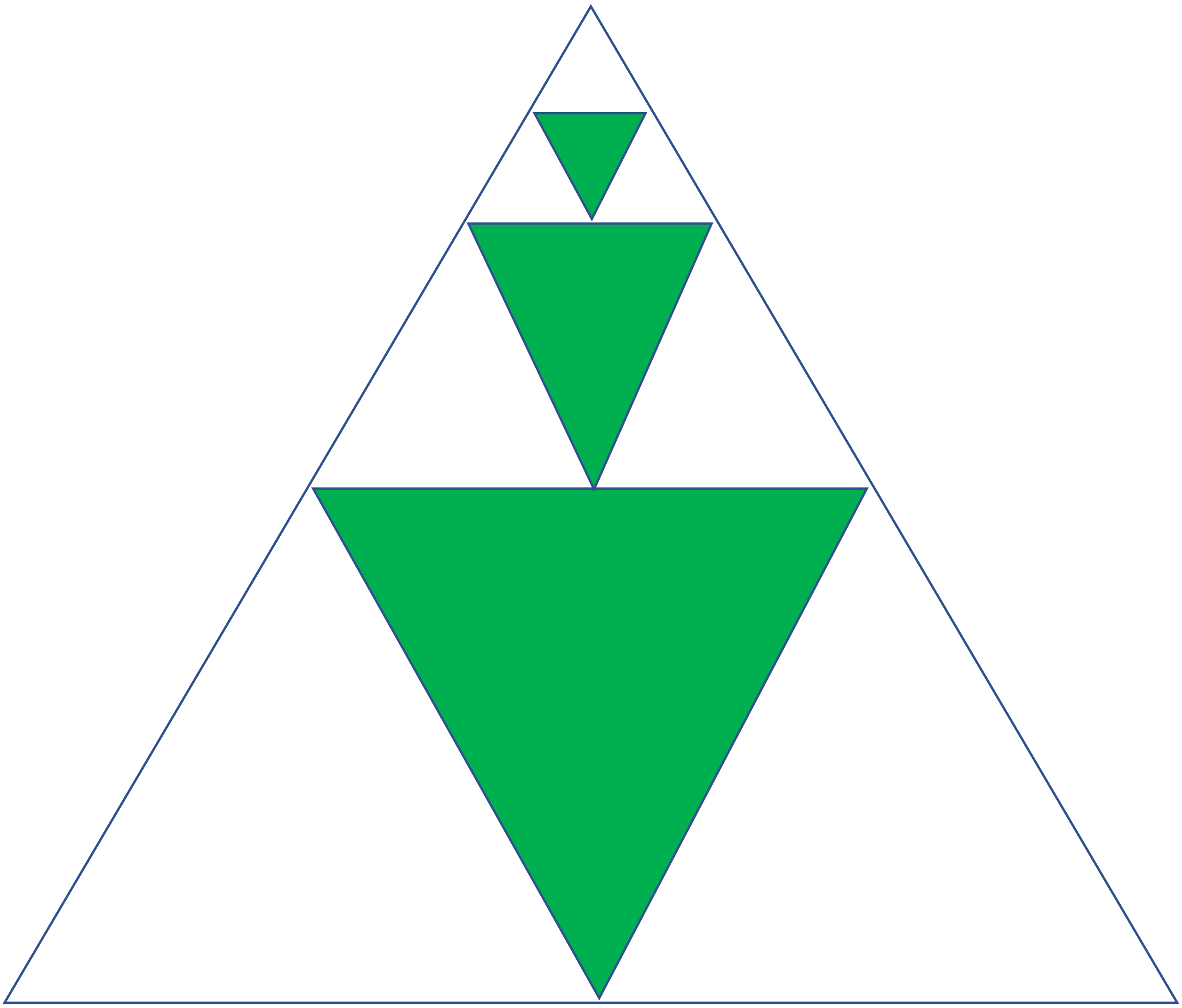
$= 8\left(\frac{1}{2}.\frac{b}{4}.\frac{h}{2}\right) = \frac{bh}{2} = \frac{1}{16}\,unit.$ $1 + \frac{1}{4} + \frac{1}{16}$

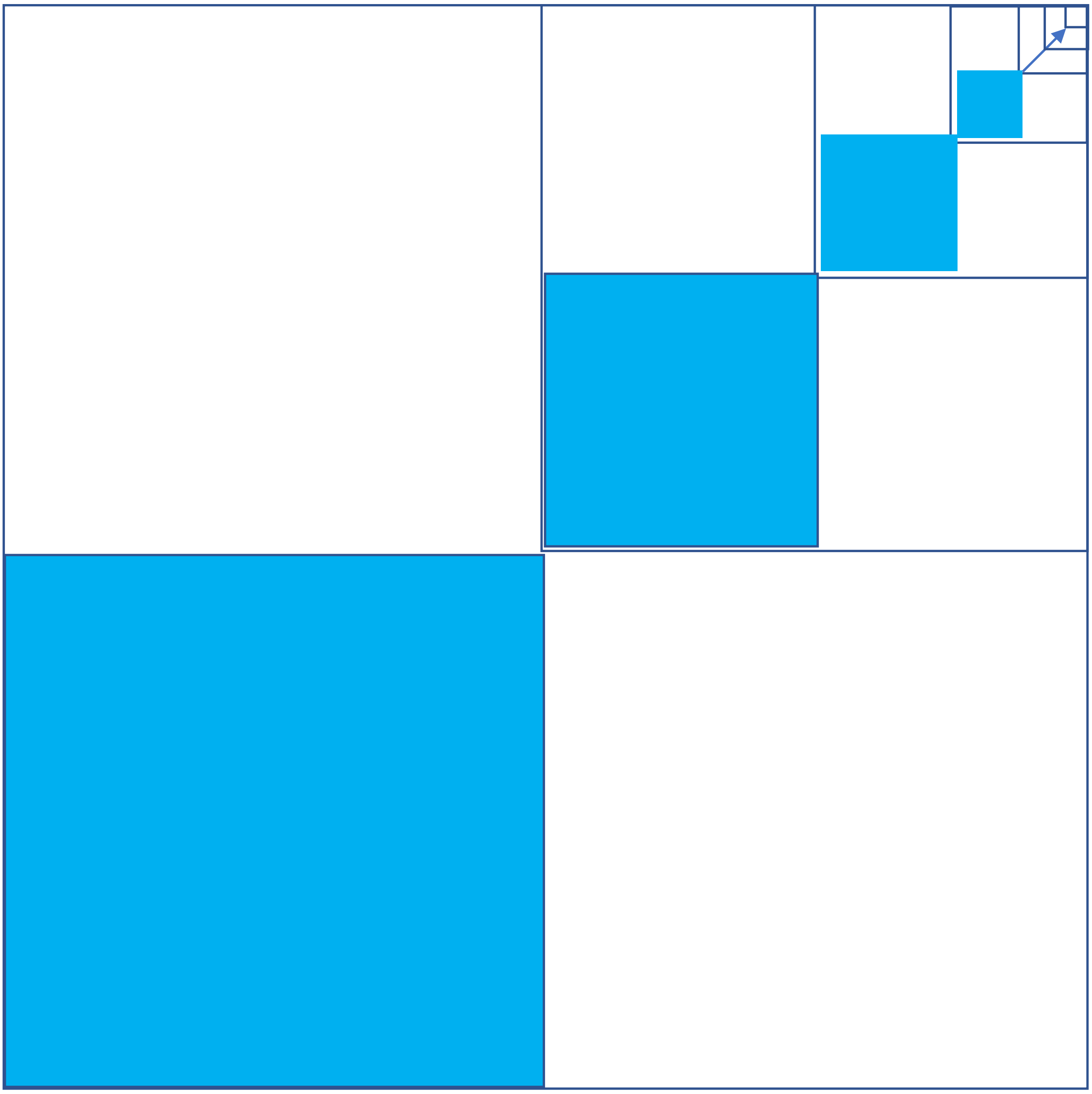
$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$

Sheet with blue squares,

Sheet with green triangles

What Archimedes needs now is a way of adding 1 + an infinite series of powers of a quarter. We shall use a trick. There are two versions of this trick. See which you prefer.





Summarise Archimedes' finding.

[The ratio of the area of the parabolic segment cut off by a line *l* is 4/3 x the area of the triangle on *l* as base and apex at the point of contact of the tangent parallel to *l*.]