Proof methods for 6403

1. Establish the LHS of the inequality as follows.
Show first that $r_A = \frac{R}{2}(1-\cos\alpha)$, whence $r_A + r_B + r_C = \frac{R}{2}[3-(\cos\alpha+\cos\beta+\cos\gamma)]$.
Substitute $\pi-(\alpha+\beta)$ for γ and simplify.
The requirement now is to maximise the expression in the round bracket.
Obtain partial derivatives of the expression with respect to α, β respectively. Set each equal to zero. Solve the resulting equation pair. You should find that the greatest value in the required interval is obtained when $\alpha = \beta = \frac{\pi}{3}$. This yields the result.
Establish the RHS of the inequality by exhausting cases. You should find that the greatest value of the sum is obtained as one angle approaches π.

2. Establish that: $\tan\alpha = \frac{a}{2(R-2r_A)}$, etc.; $a = 4\sqrt{r_A(R-r_A)}$, etc. .
Use the identity $\tan\theta + \tan\varphi + \tan\omega = \tan\theta\tan\varphi\tan\omega$, where $\theta+\varphi+\omega=\pi$.
The result follows by substitution.

Note in passing the standard identity $abc = 2(a+b+c)Rr$, which would enable us to rewrite equation **2.** In terms of *r*.

3. Establish the LHS of the inequality as follows.
Establish that by similar triangles $\frac{\rho_A}{r} = \frac{1-\sin\frac{\alpha}{2}}{1+\sin\frac{\alpha}{2}}$, etc..
For the choice made in the next line, which is not perhaps the obvious one, the author is indebted to [1].
Let φ_A be $\frac{\pi-\alpha}{4}$. Writing t_A for $\tan\varphi_A$, show that $\frac{\rho_A}{r} = {t_A}^2$, etc. **[3.1]**
The problem reduces to showing that $\frac{\rho_A+\rho_B+\rho_C}{r} = {t_A}^2+{t_B}^2+{t_C}^2 \geq 1$. **[3.2]**
Since $\varphi_A+\varphi_B+\varphi_C = \frac{\pi}{2}$, $tan(\varphi_A+\varphi_B+\varphi_C) = \frac{t_A+t_B+t_C-t_At_Bt_C}{1-t_At_B-t_Bt_C-t_Ct_A} = \infty$.
Therefore $t_At_B+t_Bt_C+t_Ct_A = 1$. [**3.3**]
Split the sum of the squares in **[3.2]** like this: $\frac{1}{2}({t_A}^2+{t_B}^2)+\frac{1}{2}({t_B}^2+{t_C}^2)+\frac{1}{2}({t_C}^2+{t_A}^2)$.
Pairing terms in **[3.2]**, **[3.3]**, you will have proved the result if you can show that
$\frac{1}{2}({t_A}^2+{t_B}^2) \geq t_At_B$, etc. .
All that is needed now is to use the classic inequality $x^2+y^2 \geq 2xy$ for $x > y$.

Establish the RHS of the inequality by exhausting cases. As in **1.**, the greatest value of the sum is obtained as one angle approaches π.

4. From **[3.1]** you have $\rho_A = {t_A}^2 r$, etc.
Use **[3.2]** with equality and the result comes out.

6. The following proof is given complete, but a shorter one may be possible.

Again, let $c = \cos\frac{\pi}{n}$. Then, from **[5.1]**:
$S_1 = 2(\tau+\sigma) = 8c$,
$S_2 = r + R = 1 + 4c + 3c^2$.

Let $c = 1-\delta$.

Then:
$S_1 = 8 - 8\delta$,
$S_2 = 8 - 10\delta + 3\delta^2$,
$S_1 - S_2 = \delta(2 - 3\delta)$. **[5.2]**

Since $n \geq 3$, $\frac{1}{2} \leq c < 1, 0 < \delta \leq \frac{1}{2}$. **[5.3]**

From **[5.3]** both factors on the right in **[5.2]** are positive, i.e. $S_1 > S_2$ and $2(\rho + \sigma) > r + R$ as required.

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