Contents

The Magic Manual

Section 4

Packings

A guide for fabricators and users to stations from the Magic Mathworks Travelling Circus

4. PACKINGS

In **4.1** and **4.3** we look at tessellations in 2-D and 3-D respectively.
In **4.5** and **4.6** we do the same for circle packings.
In **4.2** we look at tilings which do not repeat by translation.
In **4.4** we investigate an important practical packing of polyhedra.

4.1.1 TILINGS WITH REGULAR POLYGONS

c *Regular* tessellations use regular polygons of 1 kind to cover the plane. *Semi* regular tessellations use regular polygons of more than 1 kind but at every vertex the same polygons meet in the same cyclic order.

p The experimenter is required to recognise those definitions *operationally* in order to continue the tilings given. Older or more advanced students may be led to an *explicit* definition.

4.1.2 DOUBLE TILING JIGSAW PUZZLE

c Here we have 2 dissections with 4-fold rotational symmetry. With the help of an auxiliary square, each dissection itself tessellates. Thus the same pieces - a square and an irregular pentagon - produce 2 different tilings.

p In order to move easily between one tiling and the other, the puzzlist must realise how the transformation works:
the squares which form the centres of the larger squares become those between octagons;
those at the centres of octagons become those between larger squares.

4.1.3 THE TETRAHEDRON TILING

c Uniquely among the regular polyhedra, the tetrahedron can be 'rolled' - i.e. toppled about edges from face to face - so that the triangles of the tiling the 4 faces land on form 4 disjoint sets whose union covers the plane. Each set can therefore be coded by colour to produce a *chromatic* tiling.

p Here the puzzlist is challenged to abandon a purely physical process - rolling a generating tetrahedron around - in favour of a mental one - recognising patterns. (Advanced students should challenge the assertion made above and go on to investigate semiregular polyhedra.)

4.2 PENROSE TILINGS

c All the **4.1** tilings are *periodic*. These are *a*periodic. The ones here are generated from 2 rhombuses with acute angles of $\pi/10$ and $\pi/5$ respectively.

p Like all quadrilaterals, the rhombus tessellates.
If the markings on the rhombuses are ignored, a periodic tiling (or an aperiodic tiling with periodic stretches within it) is possible - and easy to accomplish.
If they are respected, a purely aperiodic tiling is possible - but difficult to achieve.

4.3 POLYHEDRAL PACKING

c In 3-D there is only 1 regular tessellation: a packing of cubes.
There is also only 1 *semi* regular example: a packing of regular tetrahedra and octahedra.

p The experimenters have both to construct the polyhedra and to pack them. They are thus forced to study not only the geometry of the packing but also that of the constituent solids. To achieve the packing they must recognise that unlike solids share a face. (Children find it easier to construct a pyramid than an octahedron from its net. Hence the proffered alternative of building an octahedron from a pair of pyramids.)

4.4 BRICK BONDS

c In **4.3** the solids meet vertex-to-vertex. Bricks walls are strong because their constituent solids (cuboids with edge ratios 6:3:2) do not. This is achieved by translating the 'courses' fractions of a brick length (halves in the case of the simplest 'bond').

p The investigators first complete a wall of a prescribed bond. They are then invited to apply the principle of construction defined in **c** to design their own bonds.

4.5 PACKING CIRCLES, SPHERES AND CYLINDERS

c Though spheres and cylinders are solids, we are here concerned only with their circular sections.
In close packing, each circle touches 6 others.

p To solve the problem posed, the visitor must achieve close packing. Help is at hand in the form of an analogue experiment - nature does the mathematics.

4.6 SPHERE PACKINGS

c The 2 closest sphere packings are presented:
hexagonal close packing, where the close-packed layers alternate in position;
cubic close packing, where the positions of the close-packed layers follow a cycle of period 3.

p The students' understanding is tested by requiring them to continue the patterns.

SECTION: PACKINGS		AGE RANGE: Appropriate point of entry - not necessarily to the task set by the caption - and levels on which extension activities generated (some to be pursued off-site)					
STATION							
NUMBER	NAME	4	7	10	13	16	19+
4.1.1	TILINGS WITH REGULAR POLYGONS	*	*	*	*	*	*
4.1.2	DOUBLE TILING JIGSAW PUZZLE			*	*		
4.1.3	THE TETRAHEDRON TILING		*	*	*		
4.2	PENROSE TILINGS			*	*		*
4.3	POLYHEDRAL PACKING			*	*	*	
4.4	BRICK BONDS			*	*		
4.5	PACKING CIRCLES, SPHERES & CYLINDERS			*	*		
4.6	SPHERE PACKINGS					*	*

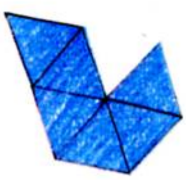
	INSTRUCTION NEEDED				SUPERVISION NEEDED			SERVICING NEEDED		
	Needs no explanation or caption	Caption enough for most people	Needs aural prompt	Visitors must be talked through stages	None	Benefits from a helper's input	Session must be directed	None	A little	Much
4.1.1		*			*			*		
4.1.2		*			*			*		
4.1.3			*		*				*	
4.2			*			*		*		
4.3		*			*			*		
4.4		*			*			*		
4.5			*		*			*		
4.6		*				*		*		

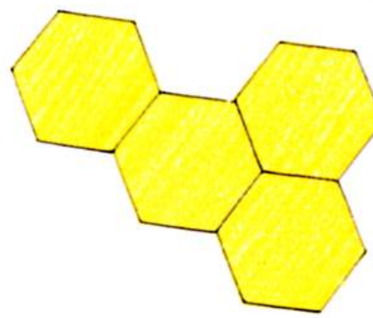
	NUMBER	TITLE
GROUP	4	PACKINGS
STATION	4.1.1	TILINGS WITH REGULAR POLYGONS
TOPIC		2-D polygonal packings without spaces (= tilings)

TILINGS

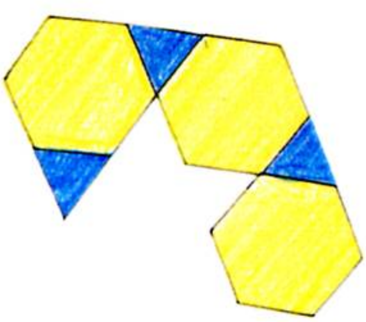
▶ **Here some tilings have been started.**

▶ **These use 1 shape:**

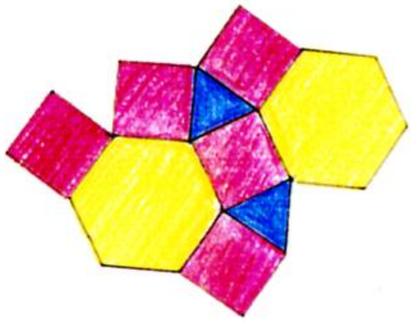




▶ **These use 2 shapes:**



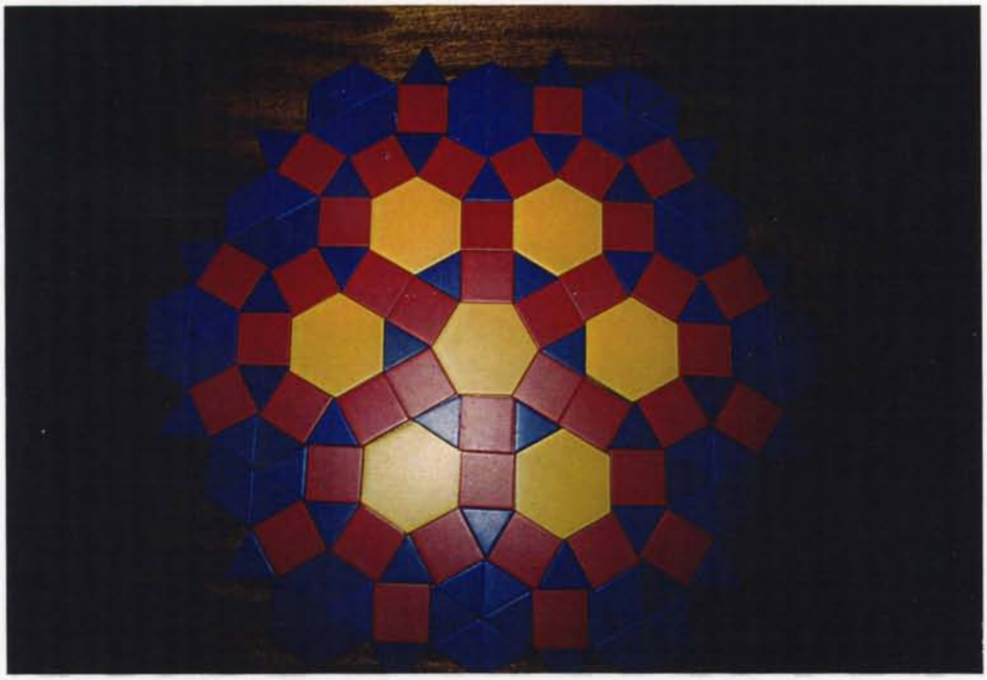
▶ **This uses 3 shapes:**



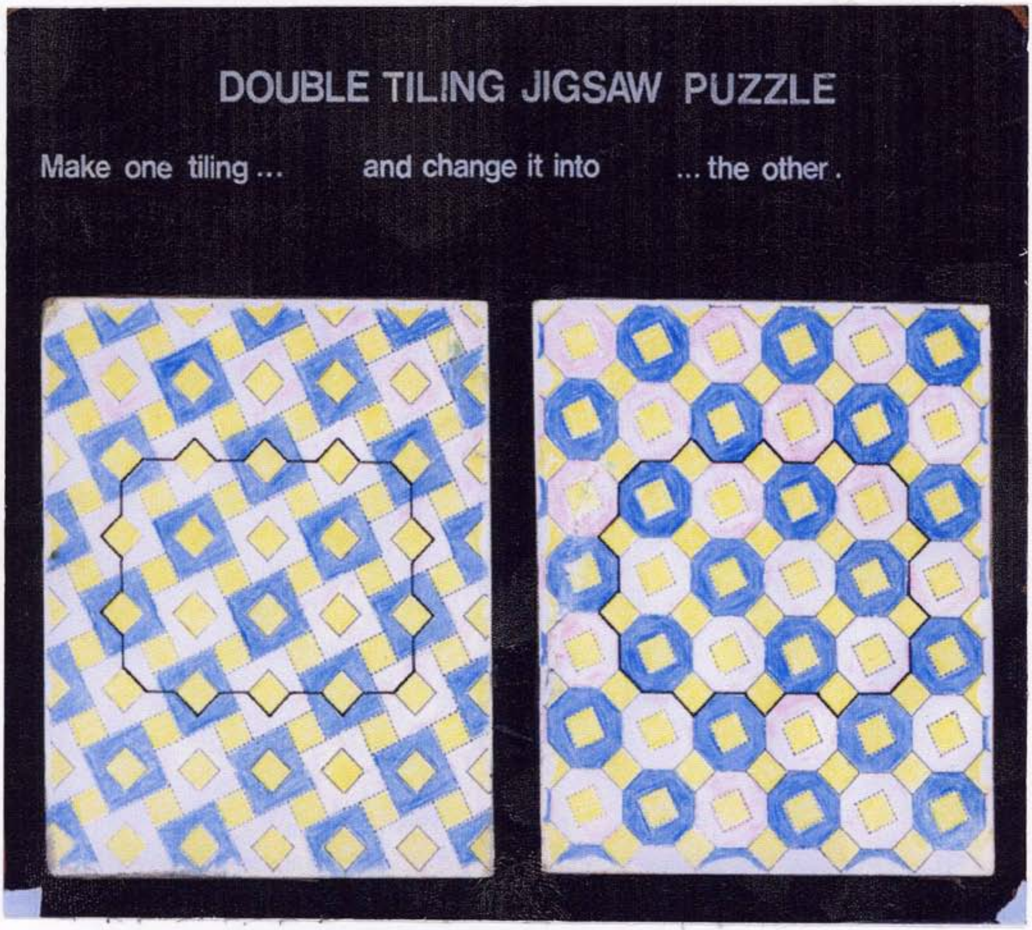
- **CARRY THEM ON.**
- **Explain to a new person how to make each one.**

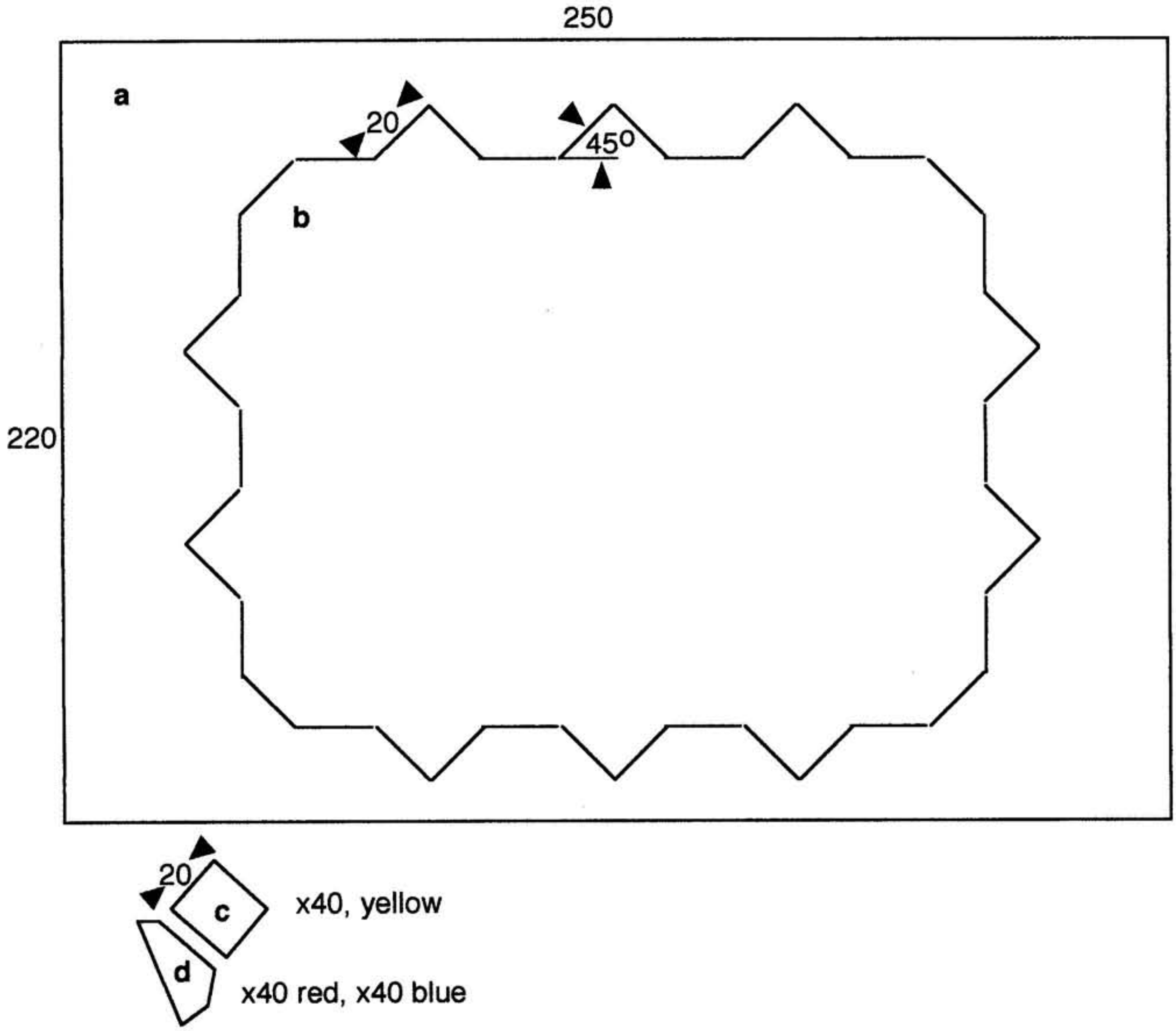

TILINGS
b
a


PACKING SHAPES

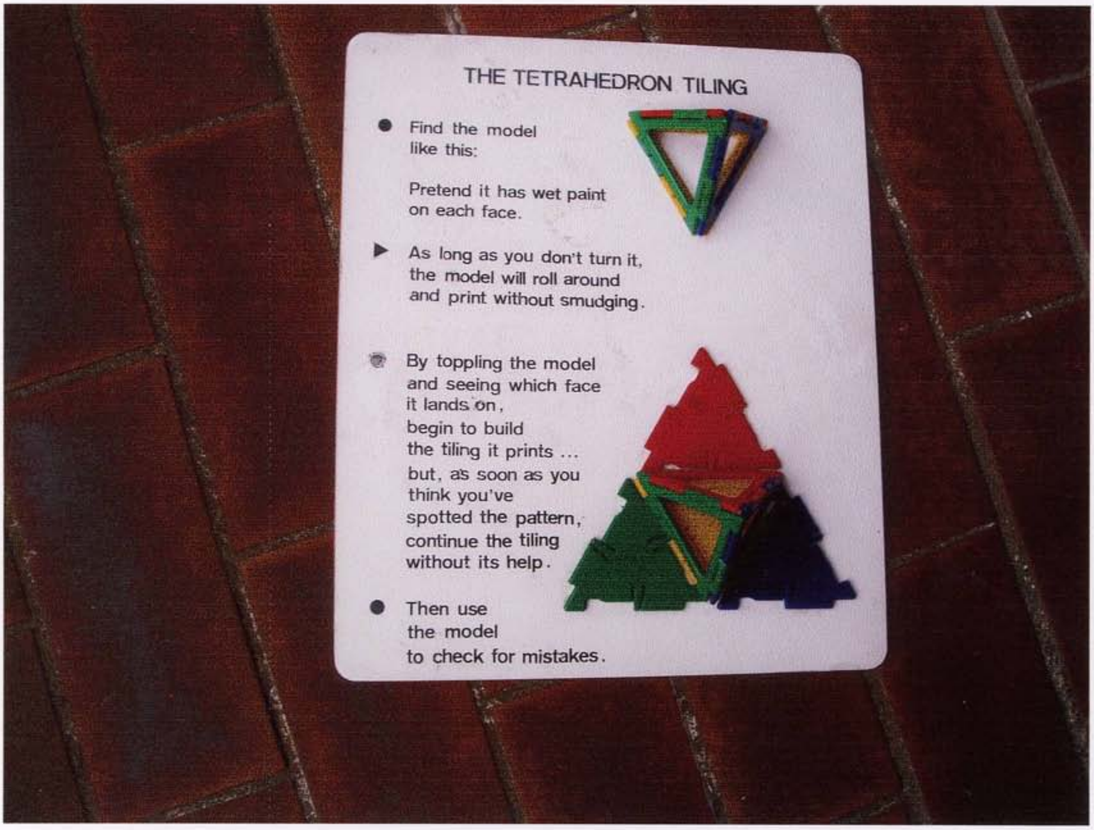


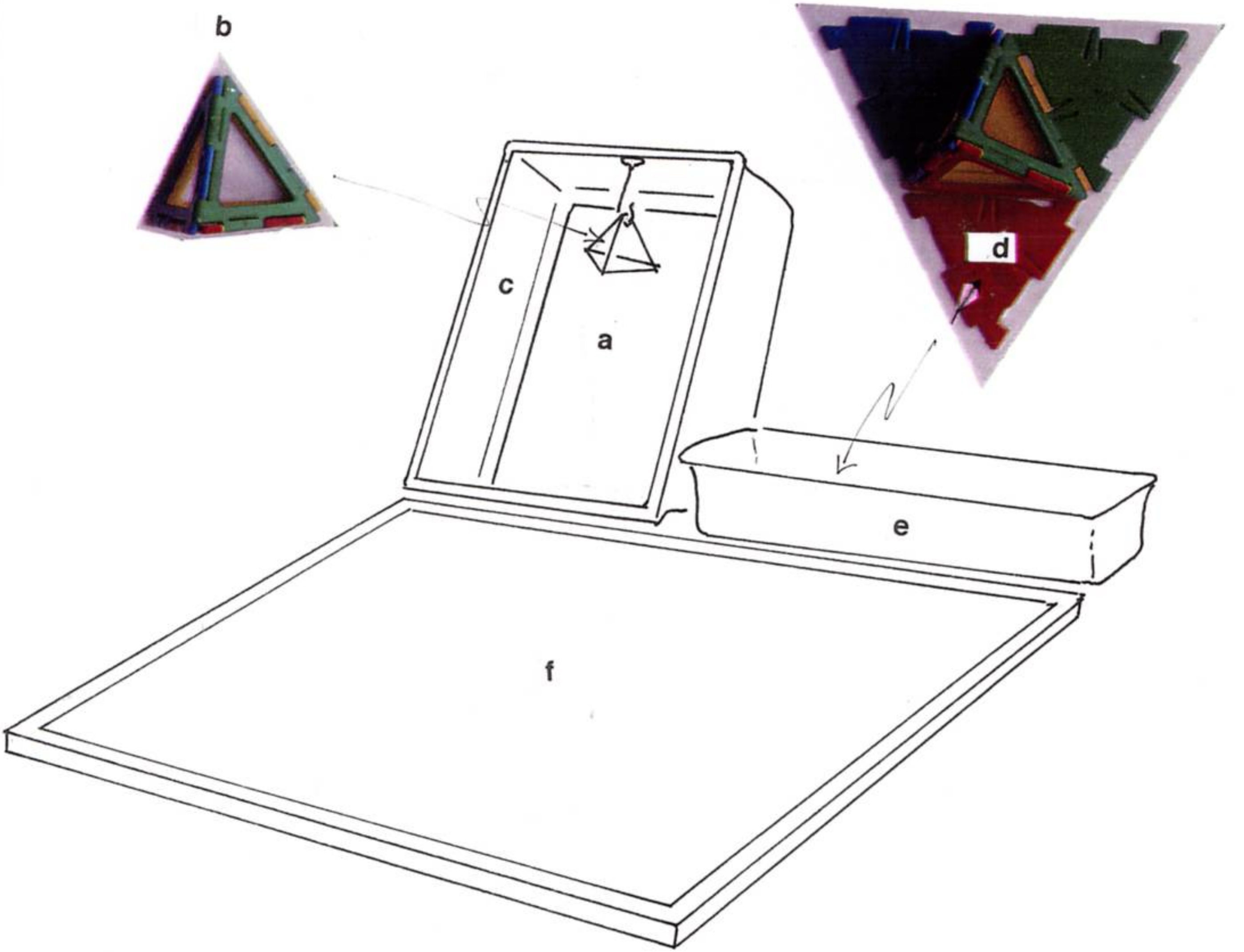
PICTURE KEY	DESCRIPTION	TRADE NAME	U.K. SOURCE
a	polygons, 3 mm PVC, 40 mm edge, die-cut by:	Amari Foam	Amari Plastics (address above) DCM Products (Coventry) Ltd Exhall Coventry CV7 9EL T +44 1203 361601 F +44 1203 367914
b	Addis Module 2000 unit 1, specimen shape stuck on 2 biggest faces		(see **THE STORAGE SYSTEM**)


DOUBLE TILING JIGSAW PUZZLE
Make one tiling ...
and change it into
... the other .


250
a
20
45o
b
220
20
c
x40, yellow
d
x40 red, x40 blue

PICTURE KEY	DESCRIPTION	TRADE NAME	U.K. SOURCE
a	PVC, black, 3 mm, stuck to:	Amarifoam	Amari Plastics
b	PVC, white, 3 mm		(For nearest depot visit web site: www.amariplastics.com)
c, d	expanded polyethylene, 3 mm **c** is a square; **d** is an irregular pentagon. ***For the dimensions of* d *consult the Magic Mathworks file 'Dissections' because the tolerances are very fine.***	Plastazote	Amari Plastics


THE TETRAHEDRON TILING
Find the model like this:
Pretend it has wet paint on each face.
As long as you don't turn it, the model will roll around and print without smudging.
By toppling the model and seeing which face it lands on, begin to build the tiling it prints ... but, as soon as you think you've spotted the pattern, continue the tiling without its help.
Then use the model to check for mistakes.

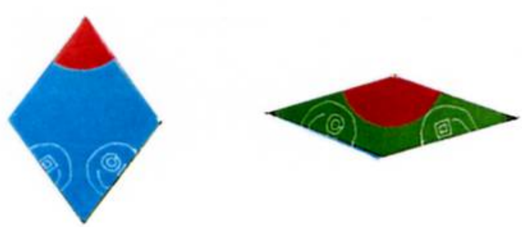

b
d
c
a
e
f

PICTURE KEY	DESCRIPTION	TRADE NAME	U.K. SOURCE
a	the caption board		
b	a regular tetrahedron in 4 colours.	Polydron Frameworks: triangles: cat. no. SKU10F300	Polydron International Ltd (address above)
c	This should be suspended from a hook in front of the caption - most conveniently by housing both in an open box **c** as shown, e.g.: This prevents it becoming lost among the other tiles.	Module 2000 System: unit 3	Addis Housewares Ltd (address above)
d	triangular tiles in 4 colours.	Polydron: triangles: cat. no. SKU100300 **2 packs needed**	Polydron International Ltd (address above)
e	tray for same, e.g.:	Module 2000 System: unit 3	Addis Housewares Ltd (address above)
f	whiteboard, 350 mm x 550 mm (or larger) to assemble tiling on		local IKEA

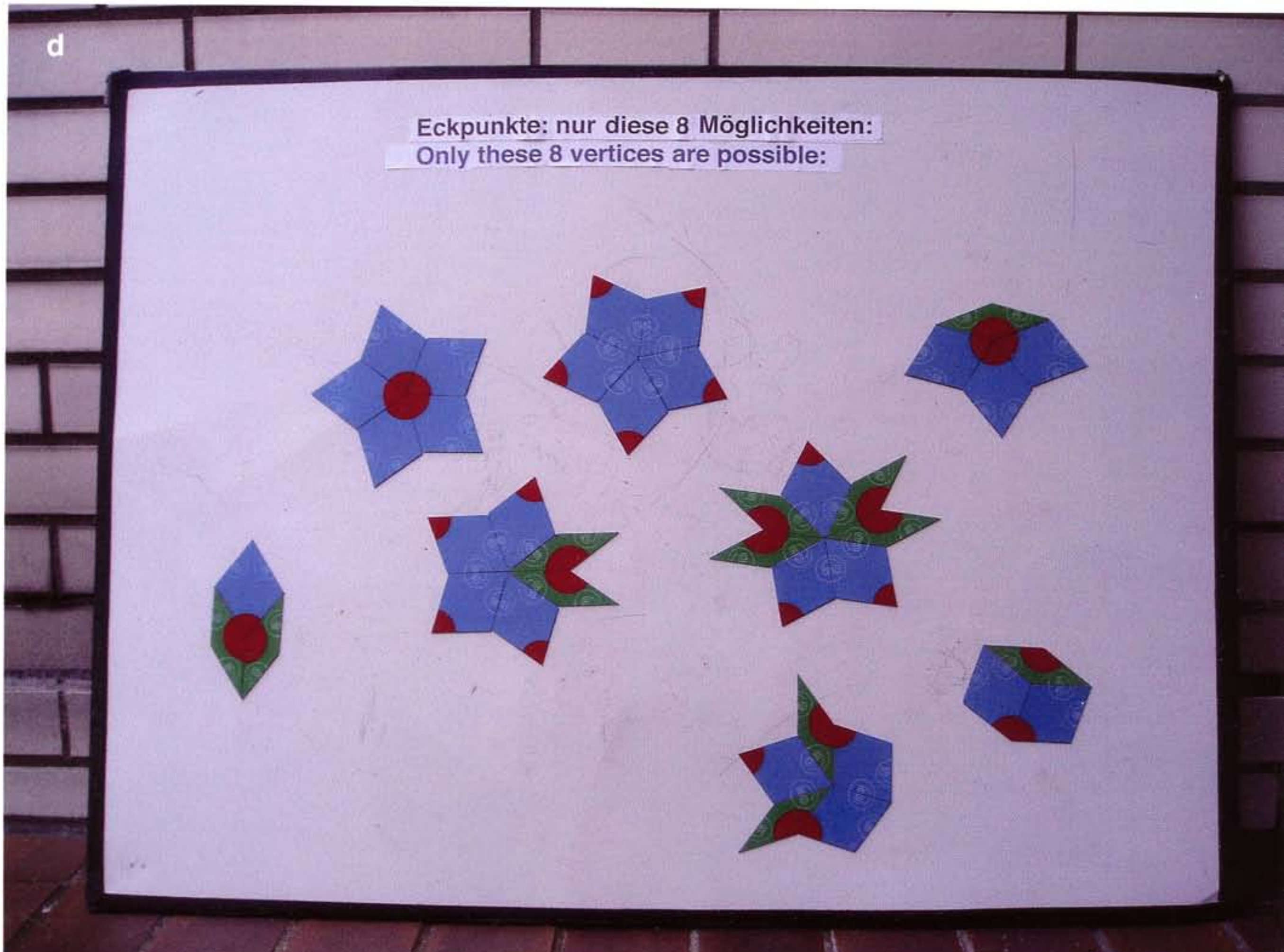
	NUMBER	TITLE
GROUP	4	PACKINGS
STATION	4.2	PENROSE TILINGS
TOPIC		Aperiodic tessellations

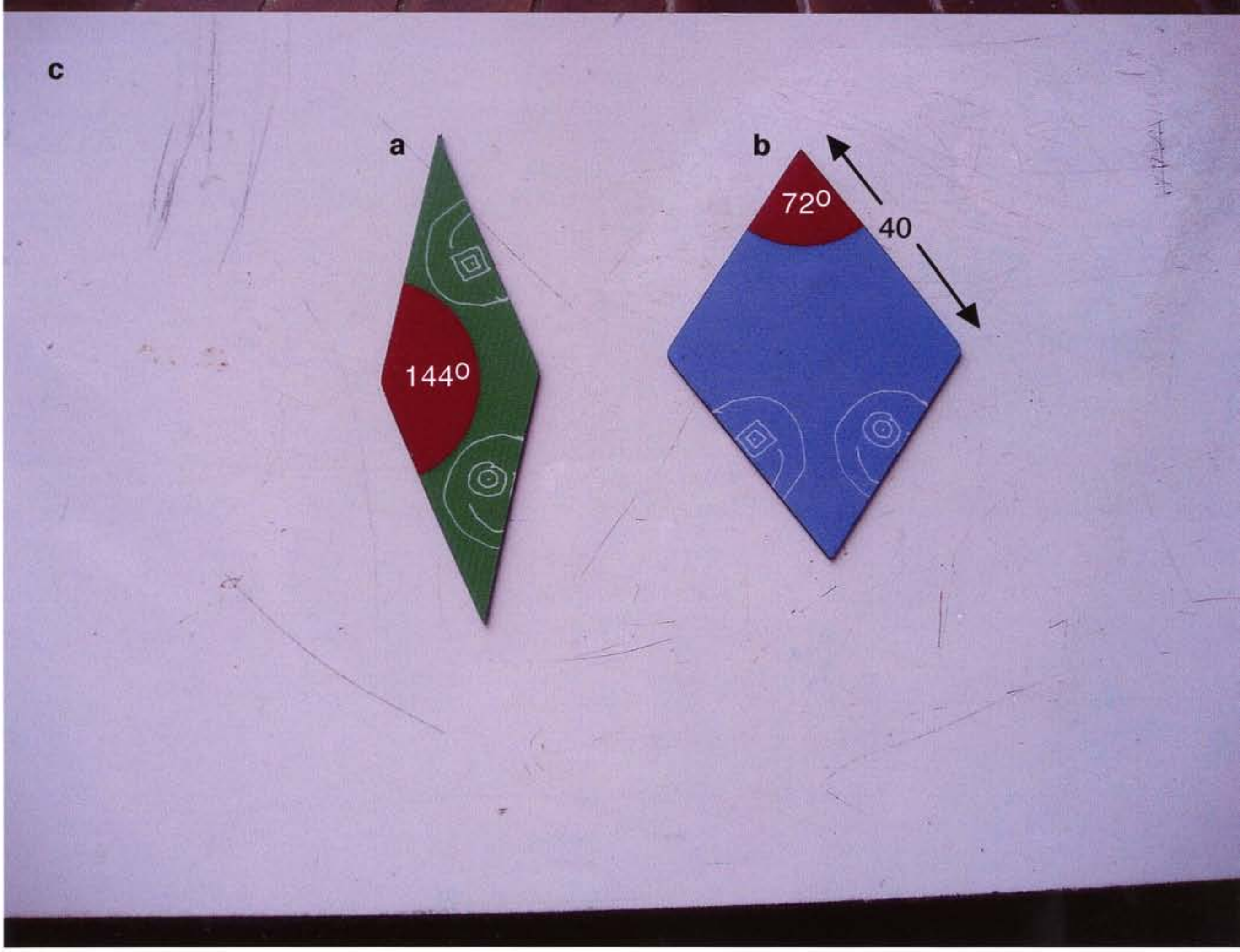
PENROSE TILINGS

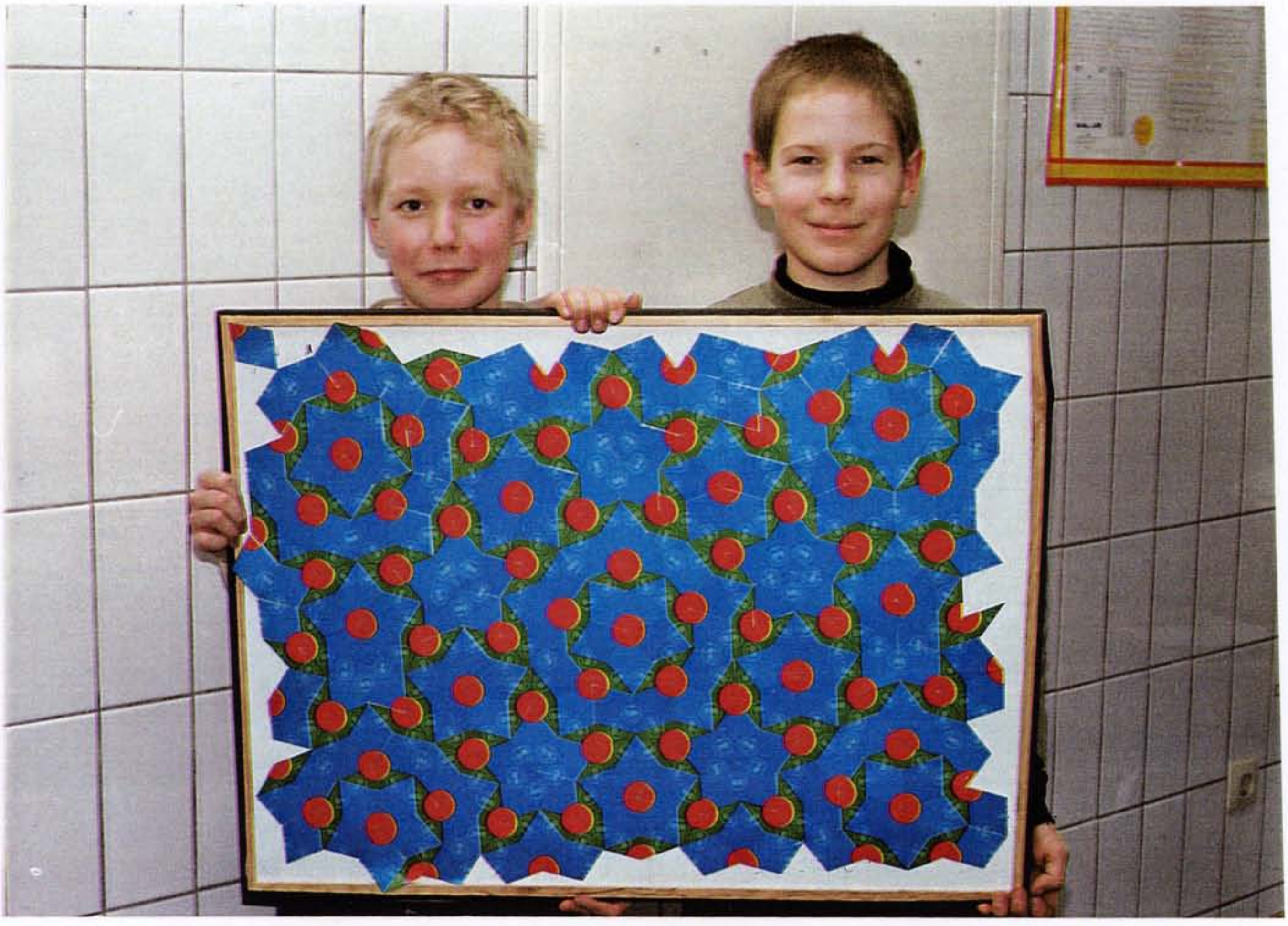
- **Making ordinary ('periodic') tilings is *easy*.**
- **Making Penrose ('*a*periodic') tilings is *hard*.**
- **You can make a Penrose tiling using these fat and thin rhombs:**



- **How many tiles can you place correctly? See how big a tiling you can make.**


d
Eckpunkte: nur diese 8 Möglichkeiten:
Only these 8 vertices are possible:


c
a
b
144°
72°
40

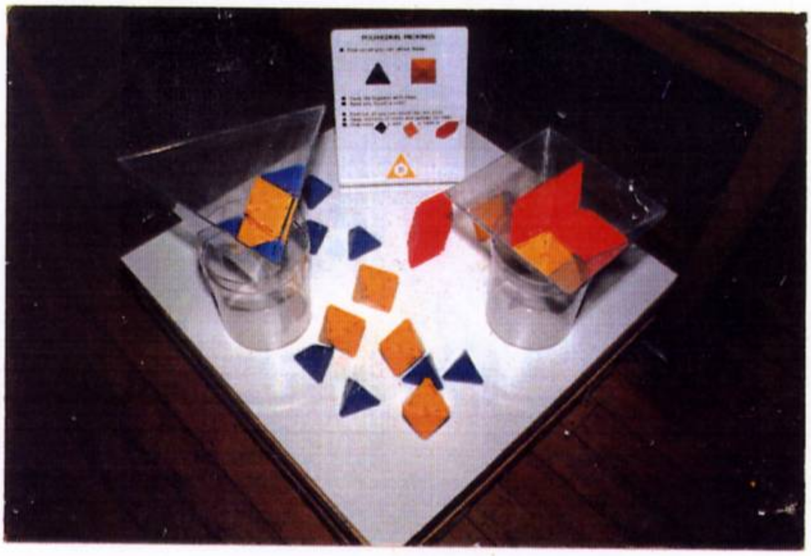


PICTURE KEY	DESCRIPTION	TRADE NAME	U.K. SOURCE
a, **b**	Penrose rhombuses, 40 mm edge, magnetic		designed by The Magic Mathworks, manufactured by: Fresco PictureWall Ltd Unit C6 Tenterfields Business Park Luddendenfoot Halifax HX2 6EQ T: +44 1422 886883 F: +44 1422 886889 E: miles@fresco.co.uk W: www.fresco.co.uk
c	magnetboard, 550 mm x 750 mm, or larger - e.g. 2 of same with frames removed joined by their longer edge to make a board 750 mm x 1100 mm		local IKEA
d	With the information on this additional board the tilers can manage larger tilings without error.		

	NUMBER	TITLE
GROUP	4	PACKINGS
STATION	4.3	POLYHEDRAL PACKING
TOPIC		The only 3-D equivalent to 4.1.1: '2-shape' case se

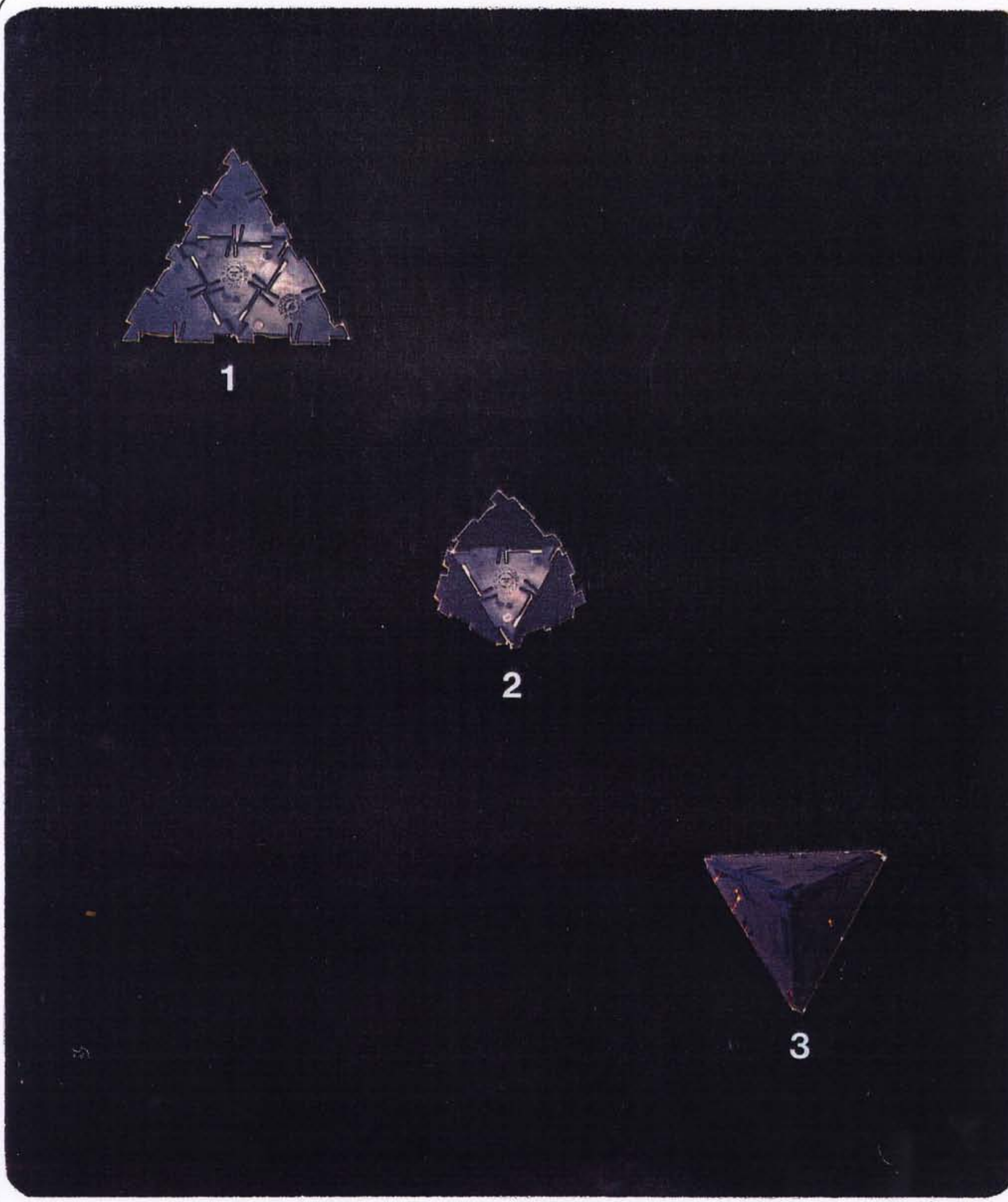
POLYHEDRAL PACKING

- Build the 3 shapes shown on the black boards - as many of each as you can.

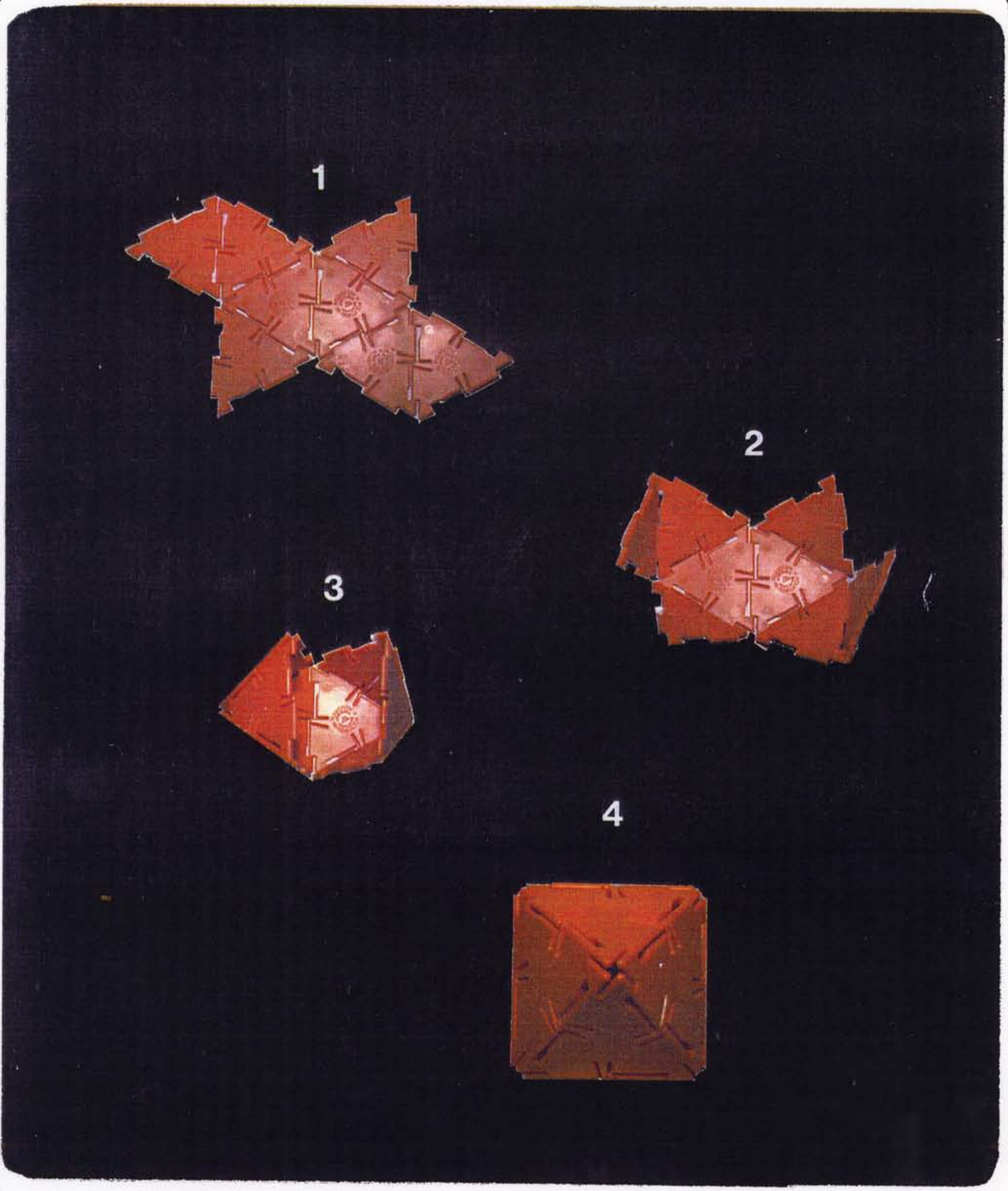


- Pack the hoppers.
  - First, use only blues and yellows.
  - Then mix in reds.

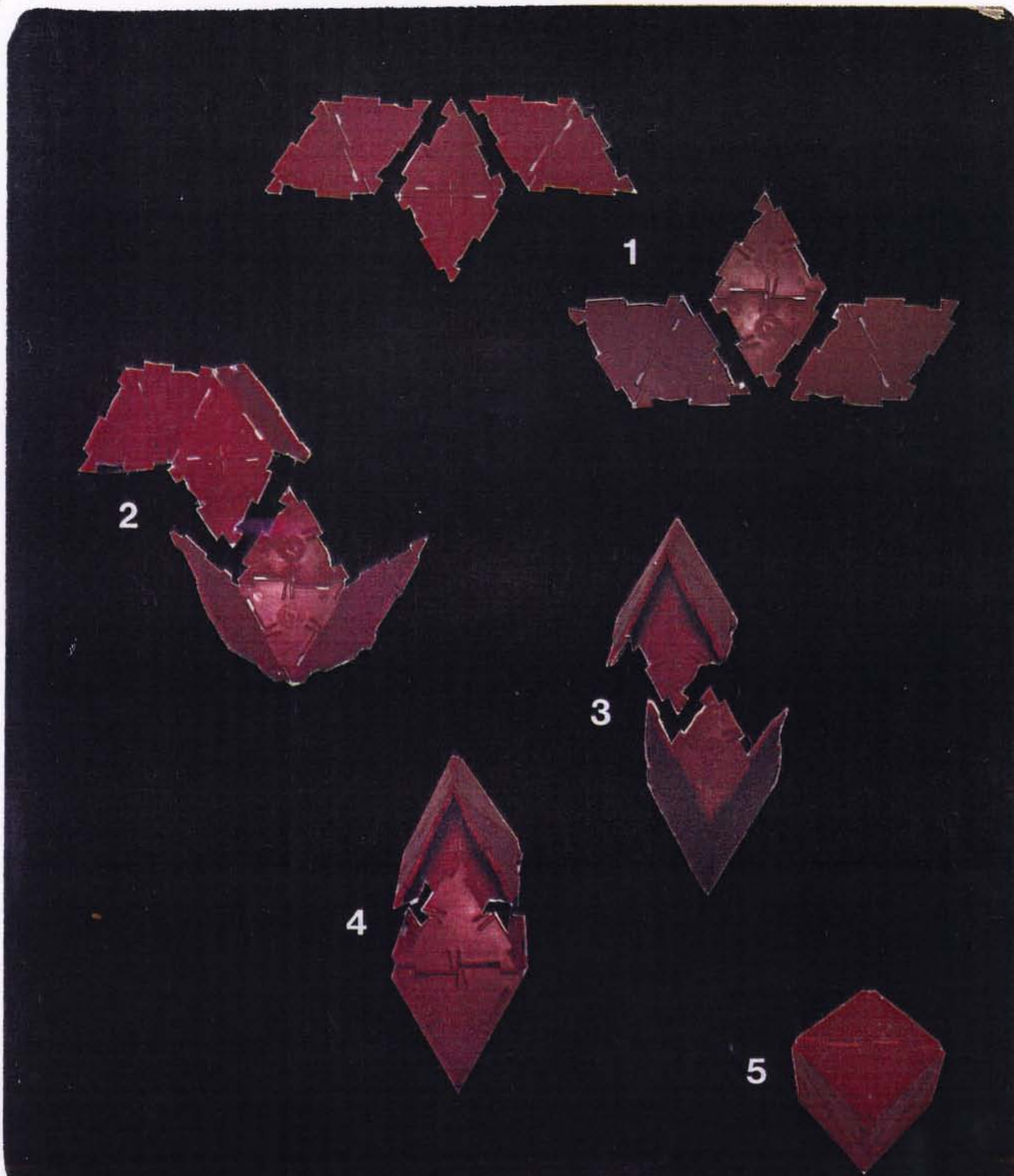
	NUMBER	TITLE
GROUP		
STATION		(Above continued)
TOPIC		

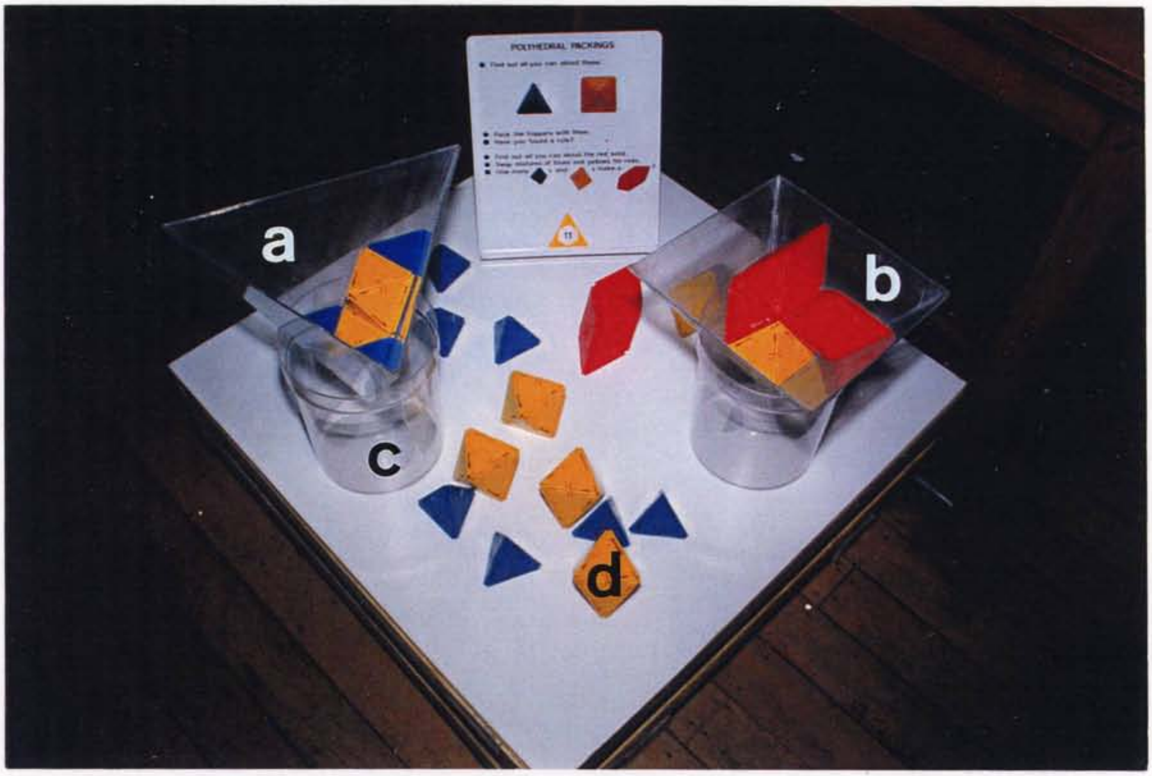


	NUMBER	TITLE
GROUP		
STATION		(Above continued)
TOPIC		



	NUMBER	TITLE
GROUP		
STATION		(Above continued)
TOPIC		



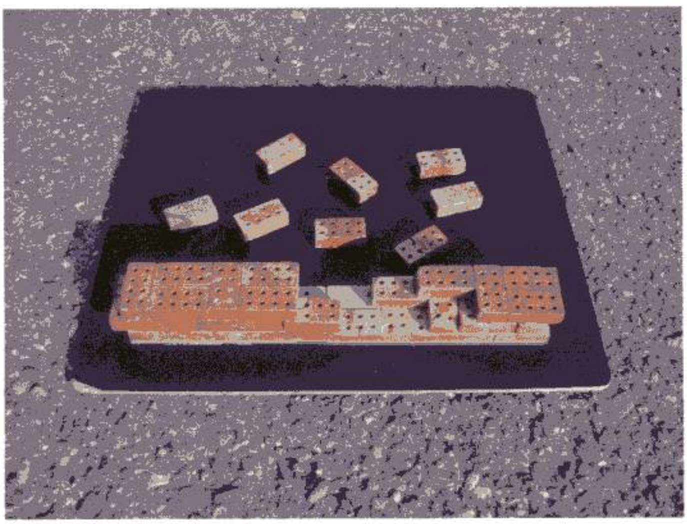

POLYHEDRAL PACKINGS
11
a
b
c
d

PICTURE KEY	DESCRIPTION	TRADE NAME	U.K. SOURCE
a	regular tetrahedron, 350 mm edge, Glodex		local
b	square-based pyramid with equilateral faces, 280 mm edge, Glodex		local
c	storage jar without lid, 200 mm < diameter < 400 mm, Perspex (or any other clear polymer), height comparable to diameter		local
d	Polydron	Equilateral triangle, cat. no. SKU100300 **12 packs needed**	Polydron International Ltd (address above)
	Note: the "black boards" are housed in glass display cases of the same area and deep enough to accommodate completed models made from the nets shown in the photographs. *In the second box an alternative to the octahedron is displayed: a dissec ion into 2 square-based pyramids .*		local

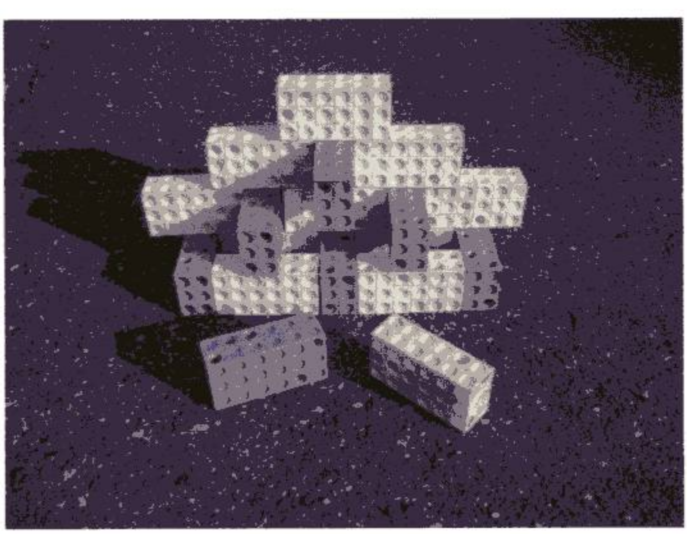
	NUMBER	TITLE
GROUP	4	PACKINGS
STATION	4.4	BRICK BONDS
TOPIC		Useful packings of cuboids

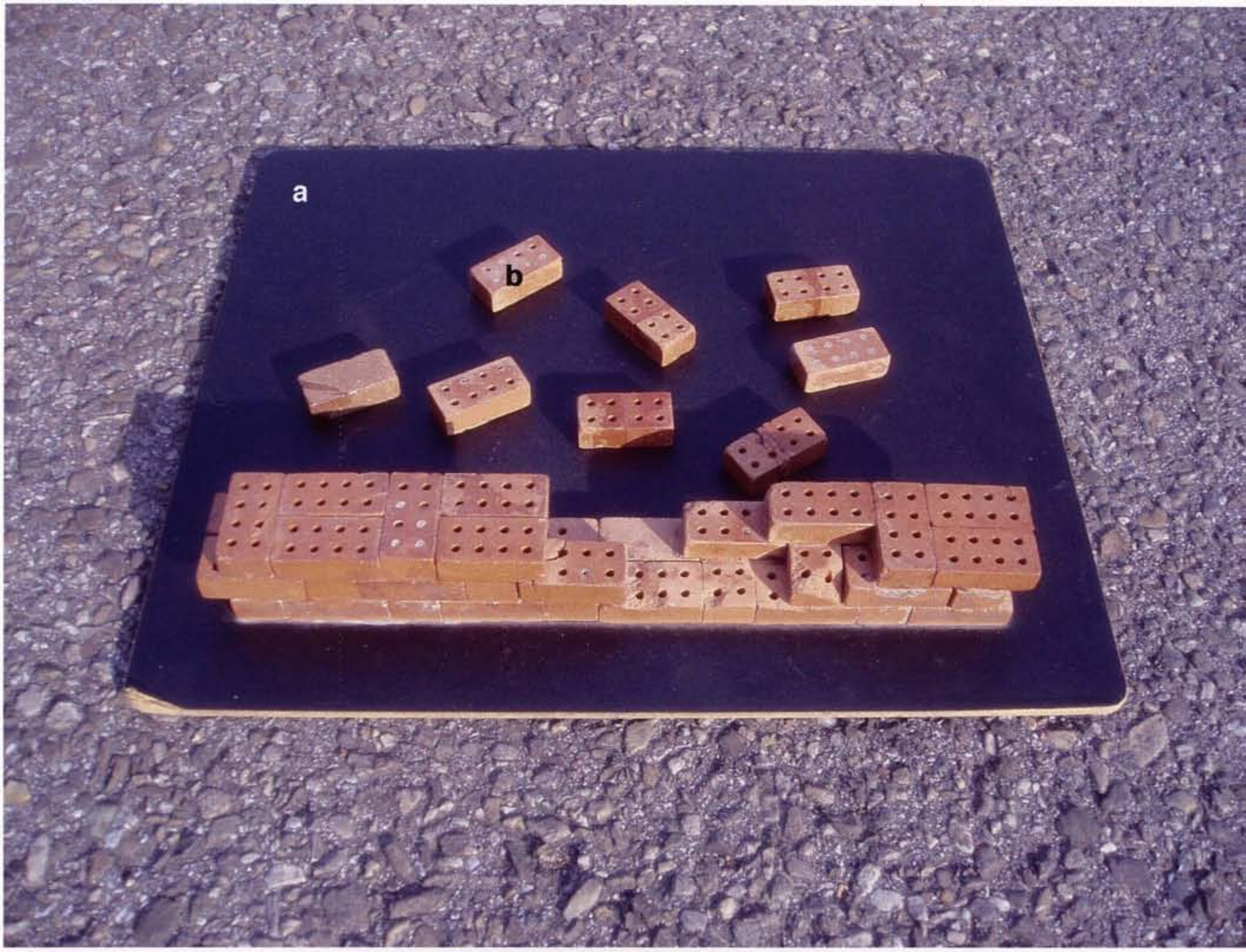
BRICK BONDS

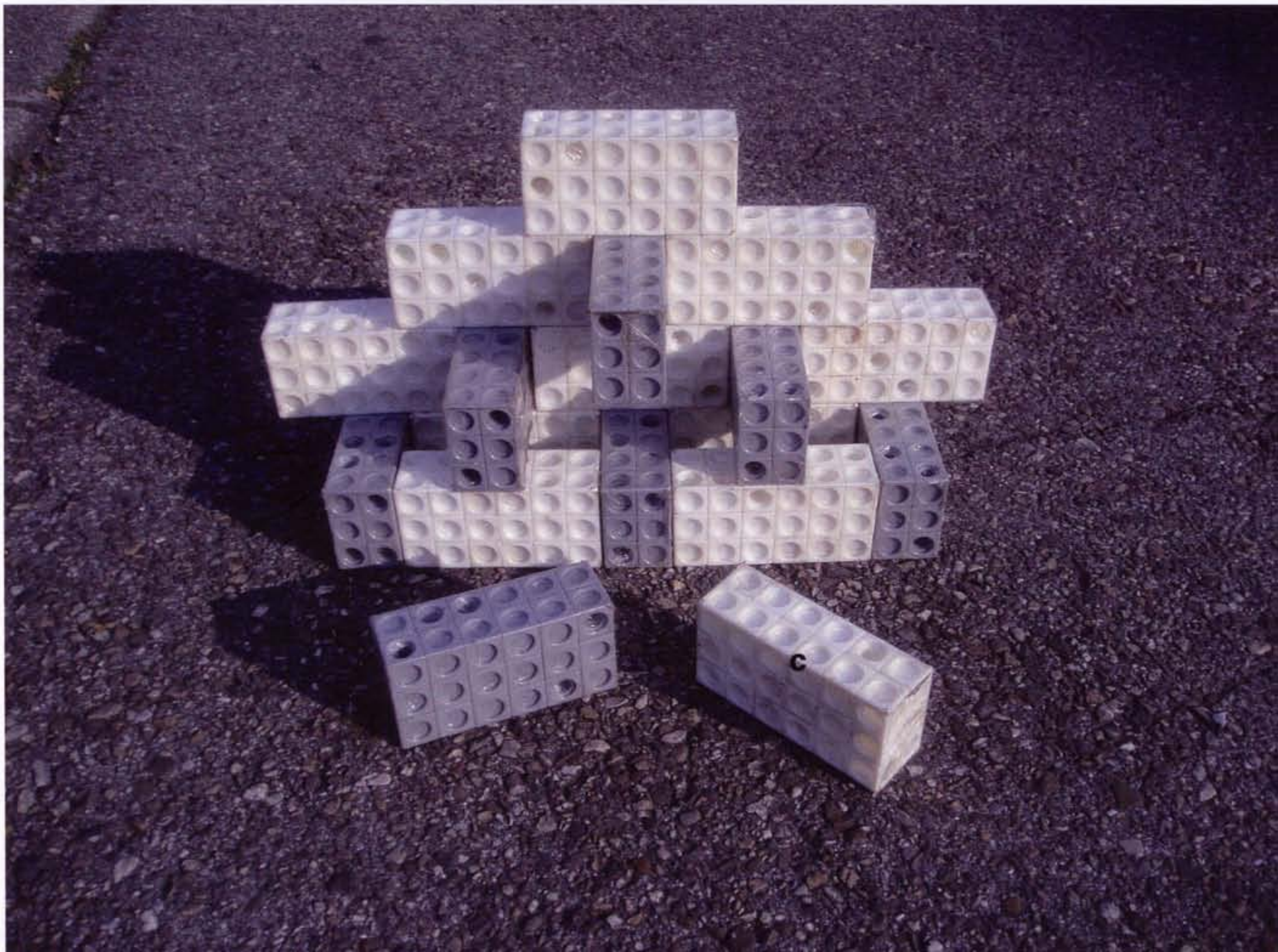
- **Brick packing patterns are called 'bonds'. The damaged wall is in 'Flemish' bond.**
- **Repair it.**



- **Use the white and grey bricks to design your own.**




a
b

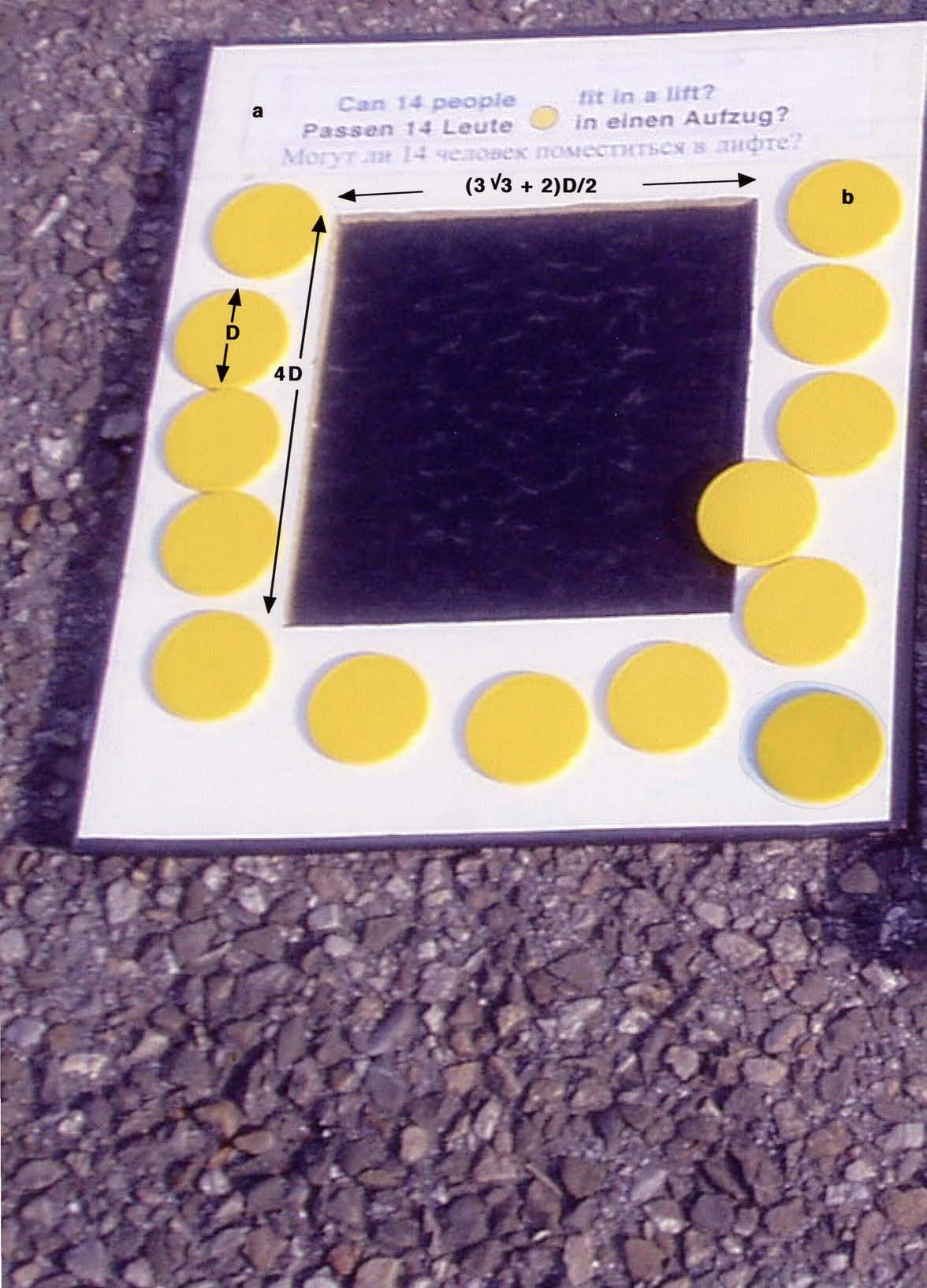

c

PICTURE KEY	DESCRIPTION	TRADE NAME	U.K. SOURCE
a	caption board as described		
b	real bricks, scaled about 1:10	Baumeister	Ton und Kugel GmbH
c	bricks from 6 x 3 x 2 = 36 interlocking cubes, 2 cm, bound with a book-covering material (e.g. Protectafilm, Transpaseal)	Multilink: NES Arnold catalogue: SY 007/9	NES Arnold Ltd (address above)

	NUMBER	TITLE
GROUP	4	PACKINGS
STATION	4.5	PACKING CIRCLES, SPHERES AND CYLINDERS
TOPIC		Packing circles and solids of circular cross-section

PACKING CIRCLES, SPHERES AND CYLINDERS

- **For the key to solving the puzzle on the black board, tilt the green tray slightly and observe the pattern the little balls make.**



a
Can 14 people fit in a lift?
Passen 14 Leute in einen Aufzug?
Могут ли 14 человек поместиться в лифте?
(3 √3 + 2)D/2
b
D
4D

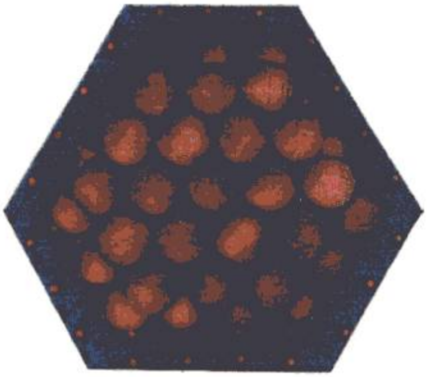

c
d
Tilt the tray.
Watch how the balls pack.
Kippe das Tablett.
KEEP THIS EDGE ON THE TABLE
e
f
g
h

PICTURE KEY	DESCRIPTION	TRADE NAME	U.K. SOURCE
a	2 caption boards as described, *stuck back-to-back*, the upper marked as shown; from the upper, a rectangular hole cut to the dimensions marked *- in this case D = 40 mm*		
b	counter (yellow), 40 mm diameter		local
c	tray		local/(school suuplier)
d	insert, marked as shown The longer dimension of the space remaining should be a multiple of the diameter of the balls, **e**, so that they line up neatly along the lower edge when the tray is tilted - or fit into spaces in the *second* row so aligned.		
e	wooden balls, 10 mm diameter		local
f	*Glodex* cover to contain balls, supported by...		local
g	pillars, slightly higher than 10 mm		local
h	*The 'lift' - the rectangular hole in* **a** - is appropriately scaled down (here 4:1) and marked in correction fluid.		

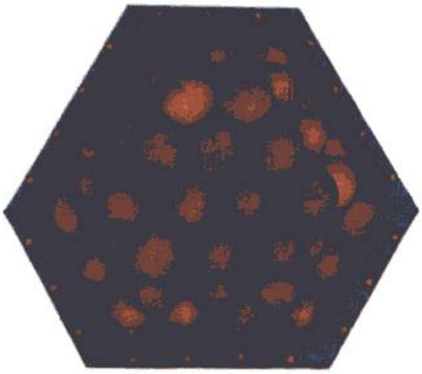
	NUMBER	TITLE
GROUP	4	PACKINGS
STATION	4.6	SPHERE PACKINGS
TOPIC		The 2 close-packed arrangements of spheres

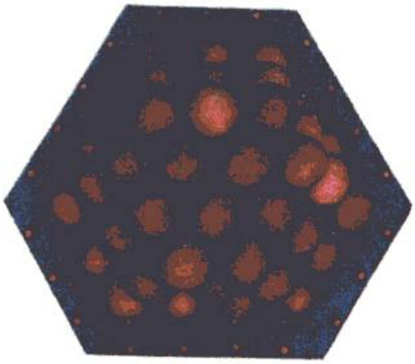
SPHERE PACKINGS

- Build a model 2 layers deep:



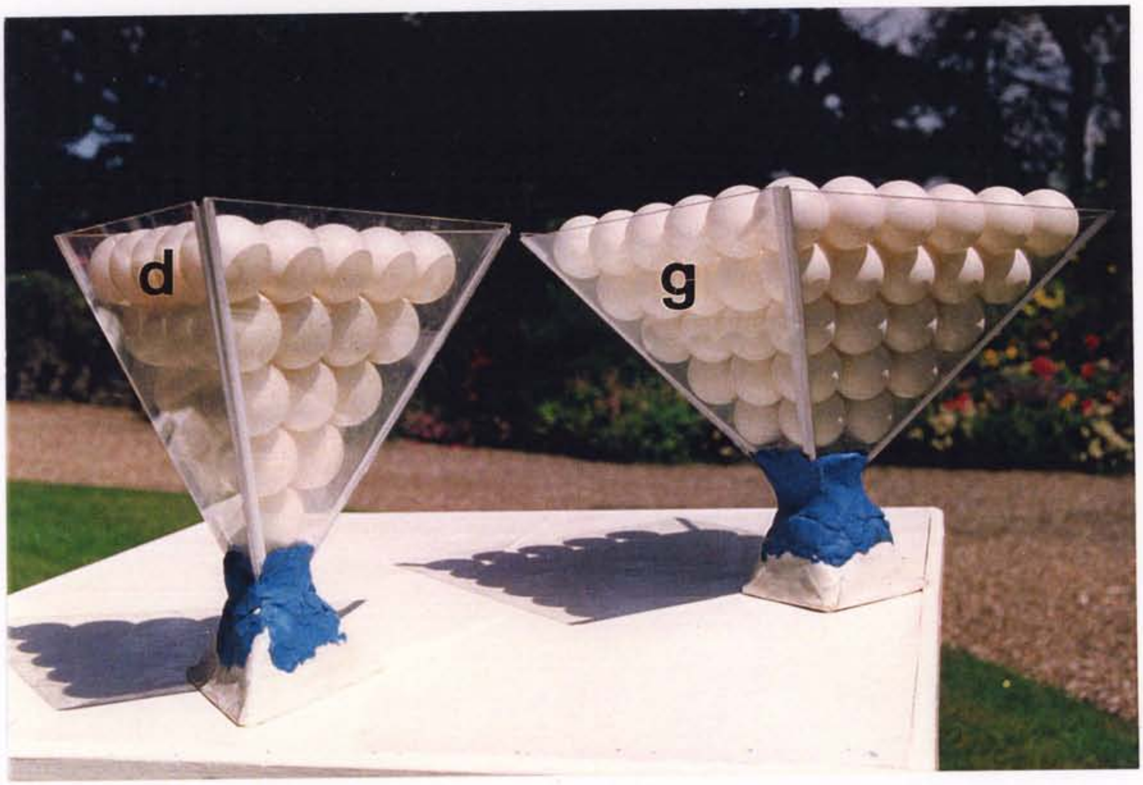
- Look down at layer 1 through the gaps in layer 2. Follow ... either rule B ... or rule G.
  For layer 3, place a ball above every ... ball in layer 1 (rule B): ... gap in layer 1 (rule G):

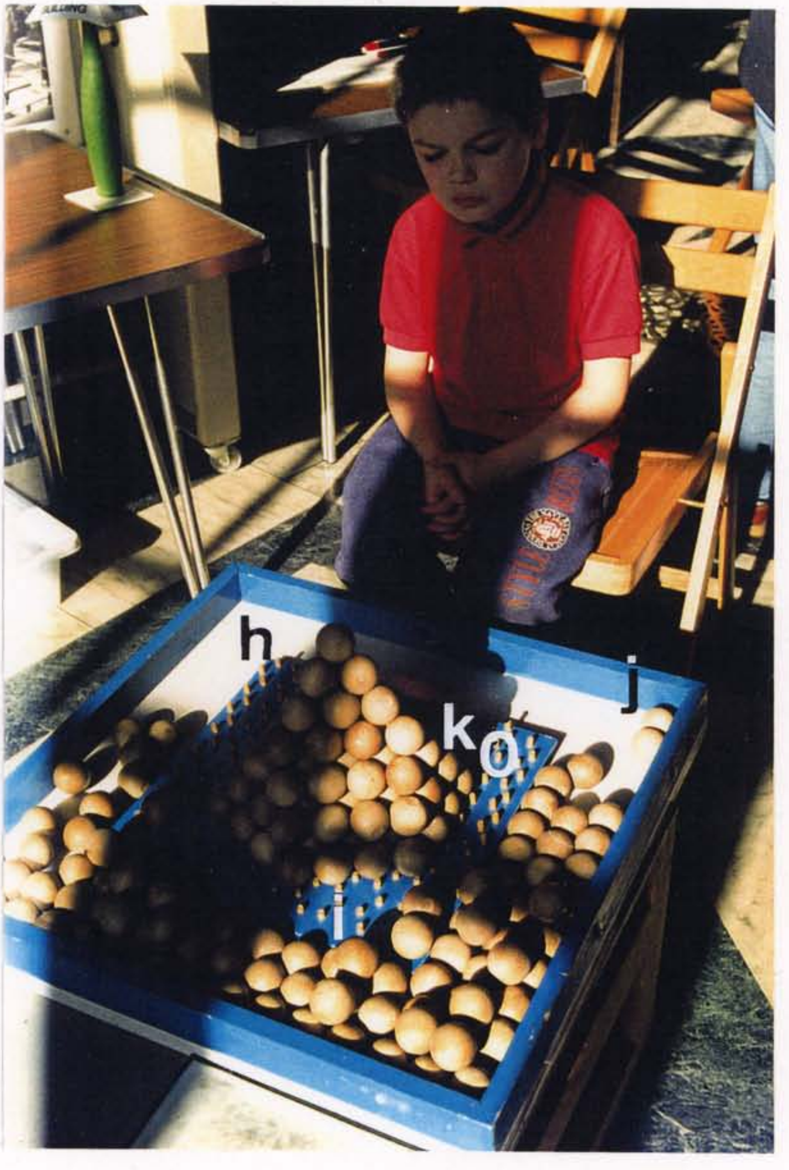




- Now use the whole board. Build as high as you can. Use ... either rule B ... or rule G.


a
b
e
c
f


d
g


h
j
k
o
i

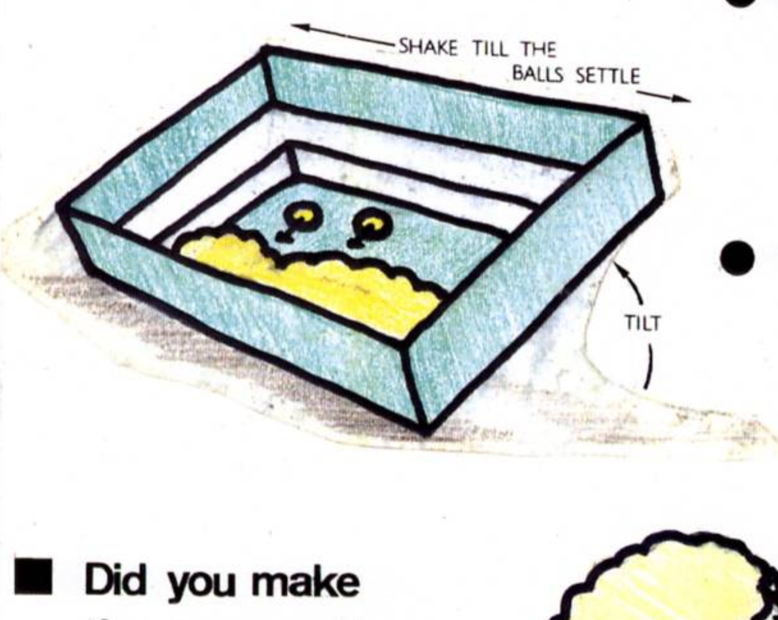
PICTURE KEY	DESCRIPTION	TRADE NAME	U.K. SOURCE
a	wood balls, 35 mm diameter		local
b,c,d	In these 3 variants horizontal layers are close-packed, so also are layers parallel to exposed faces in **b**, layers parallel to the container walls in **c** and **d**.		
e,f,g	In these 3 variants the close-packed layers are those parallel to exposed faces in **e**, those parallel to the container walls in **f** and **g**. **c** = **4.2.a** + **4.2.c** with wood balls. **d** = **4.2.a**, base attached, with table tennis balls. **f** = **4.2.b** + **4.2.c** with wood balls. **g** = **4.2.b**, base attached, with table tennis balls.		
b,e	<u>common details</u>		
h	20 mm medium-density fibreboard (MDF), 600 mm x 600 mm		local
i	20 mm MDF, 375 mm x 375 mm		local
j	wood sold in the U.K. as 'two-by-one, planed all round': $1\frac{3}{4}$" (44 mm) x $\frac{3}{4}$" (18 mm)		local
k	$\frac{1}{4}$" (6 mm) wood dowel		local
b,e	<u>differences</u> colour-coded as shown to distinguish peg arrangements as follows:		
b	The pegs form a hexagonal grid, derived from an isometric grid of edge 20 mm, i.e. $\sqrt{3}/3$ x ball diameter, set at 30° to the board edge. They should extend about 15 mm above the board surface and be sunk into it a comparable distance.		
e	The pegs form a square grid of edge 25 mm, i.e. $\sqrt{2}/2$ x ball diameter, set at 45° to the board edge. They should extend 7.5 mm above the board surface and be sunk into it a comparable distance. This figure must be calculated from the formula $h = b - \sqrt{d(2b - d)}$, b = ball radius, d = dowel radius, h = height required		

	NUMBER	TITLE
GROUP	4	PACKINGS
STATION	4.5	PACKING CIRCLES, SPHERES AND CYLINDERS
TOPIC		Packing circles & solids of circular cross-section

PACKING CIRCLES, SPHERES AND CYLINDERS

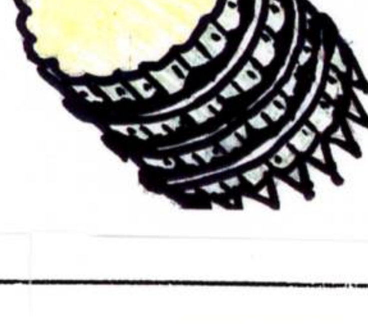
- Make this pattern with the disks: →

- How many touch each one?
- Pack the disks more tightly. Make the gaps  smaller.
- How many disks touch each one now?



- Use the ping-pong tray to show you the tightest packing of spheres.
- Use the pencil pack bound with rubber bands to show you the tightest packing of cylinders.

- Did you make the same pattern with the disks?




SPHERES AND CYLINDERS
g
f
a
b
c
d
e
SHAKE

PICTURE KEY	DESCRIPTION	TRADE NAME	U.K. SOURCE
a	counters, 37-8 mm diameter		local
b	PVC, 3 mm, 300 mm square	Amari Foam, 3 mm	Amari Plastics (address above)
c	table tennis balls (37.5 mm diameter)		local
d	tray: that illustrated a discontinued line; nearest equivalent:	Stack'n' Store	
		NES Arnold catalogue: green: NB 4204/28	NES Arnold Ltd (address above)
e	Insert such a piece only if the internal tray length is not an exact multiple of a ball diameter.		
f	pencil crayons of circular section, bound with rubber bands		local
g	caption board as described		