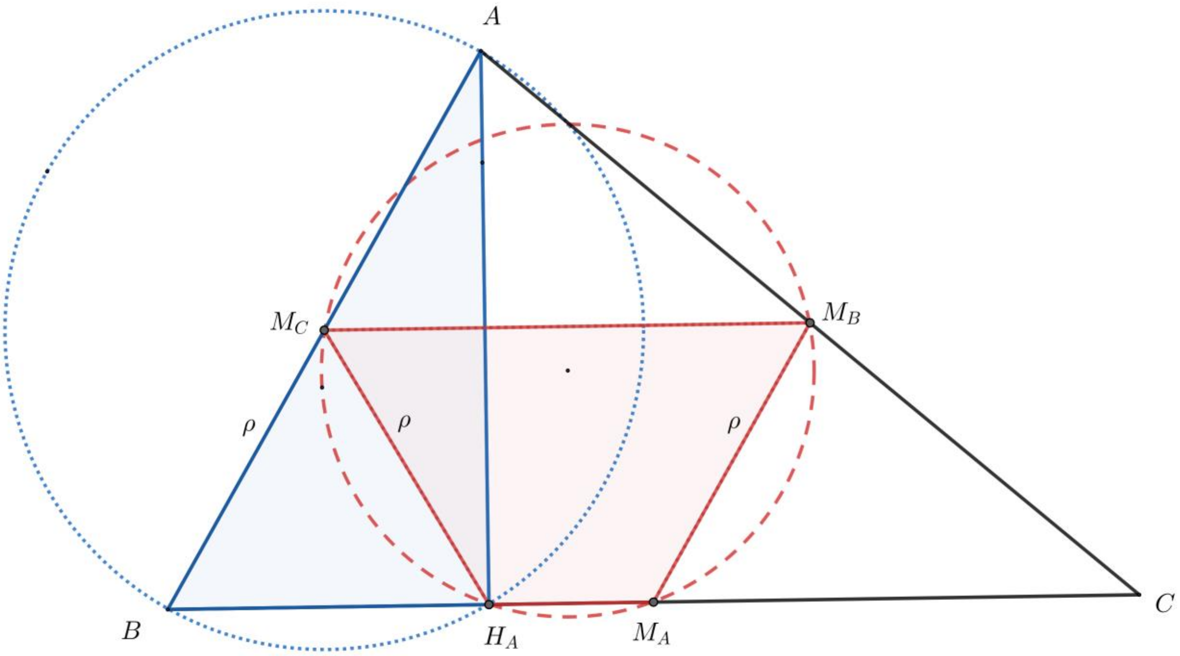
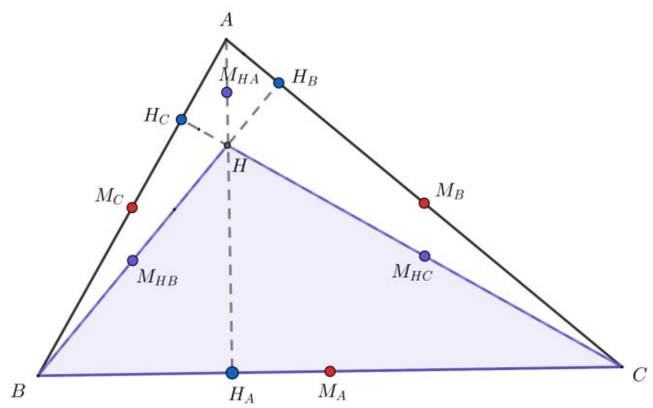
3.5 The nine-point circle

Three points define a circle. Thus, if two circles share three or more points, they are the same circle.



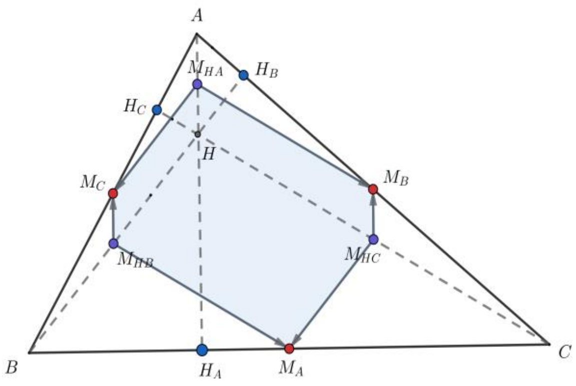
M_A, M_B, M_C are the midpoints of the sides of triangle *ABC*. Since the blue triangle has a side as diameter, the angle at H_A is a right angle and H_A the foot of the altitude from *A*. M_CH_A is a radius of the blue circle, which is also equal to M_AM_B since this is half the length of *AB*. Since M_CM_B is parallel to H_AM_A, the red figure is a regular trapezium. Since regular trapezia are cyclic, the three midpoints and H_A are concyclic. The same goes for the feet of the other altitudes, so we have a 6-point circle containing the three mid-points and the three feet of the altitudes.

Now consider triangle *HBC*. *H* is the orthocentre of triangle *ABC*. M_{HB}, etc., are the so-called Euler points. These are the midpoints of the altitude segments lying between the orthocentre and the vertices.

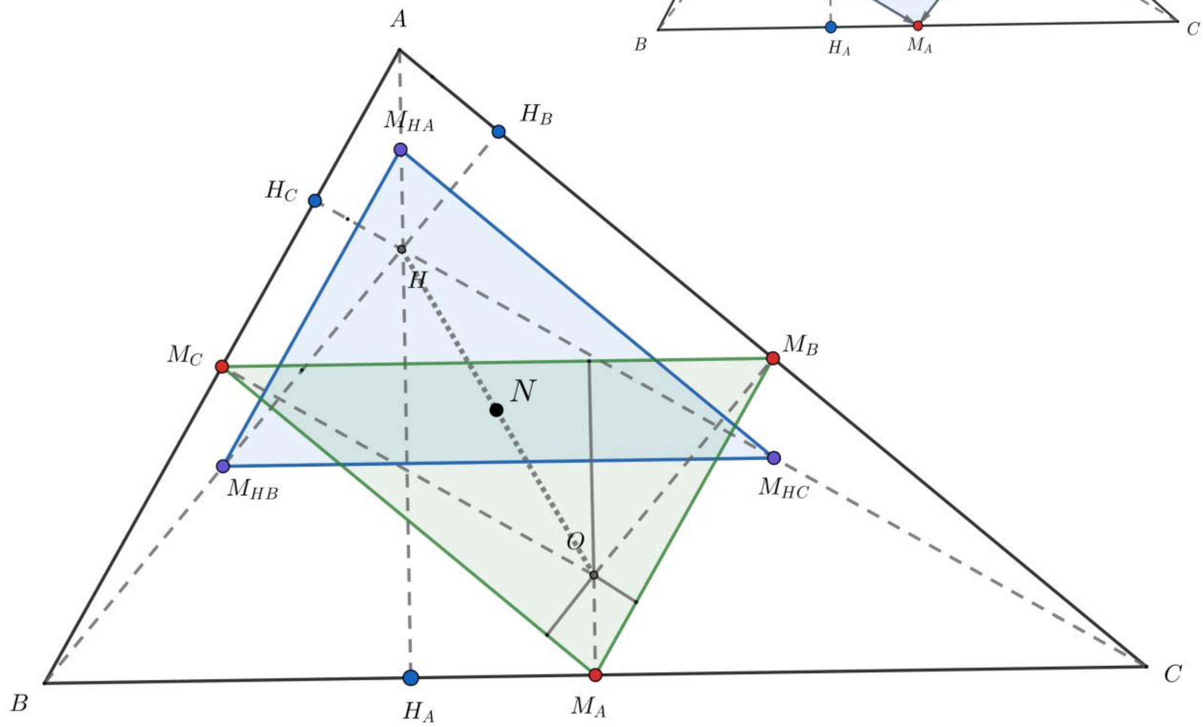


The feet of the altitudes are the same as those of triangle *ABC*, but the midpoints of the sides are now M_A, M_{HB}, M_{HC}.
So this gives us the 6-point circle for triangle *HBC*. Taking triangles *HCA*, *HAB* in turn, we have three 6-point circles, any two of which share 4 points. Therefore the three midpoints, the three altitude feet and the three Euler points constitute a circle of 9 points.

The vectors here are parallel to the altitudes. They have half the length of the segments between the orthocentre and the vertices. Thus the blue hexagon is a zonogon, a polygon with opposite sides equal and parallel.



Where does the centre of the 9-point circle lie?



The blue triangle connects the Euler points. The green triangle connects the side midpoints. Both triangles are similar to triangle *ABC*, half the size, mutually rotated a half turn. *H* is the orthocentre of the blue triangle. *O*, the circumcentre of triangle *ABC*, is the orthocentre of the green triangle. So we have a hexagon with half-turn symmetry lying on the 9-point circle, and two points within it, *H* and *O*, symmetrically disposed about the hexagon's centre of symmetry and therefore also about the centre of the 9-point circle. That is, the centre of the 9-point circle, *N*, lies at the midpoint of the Euler line.

In the section **The Euler line** we establish that *G* divides *OH* in the ratio 1 : 2. We now know that *N* divides *OH* in the ratio 1 : 1. The 4 points therefore divide the line into segments standing in these ratios:

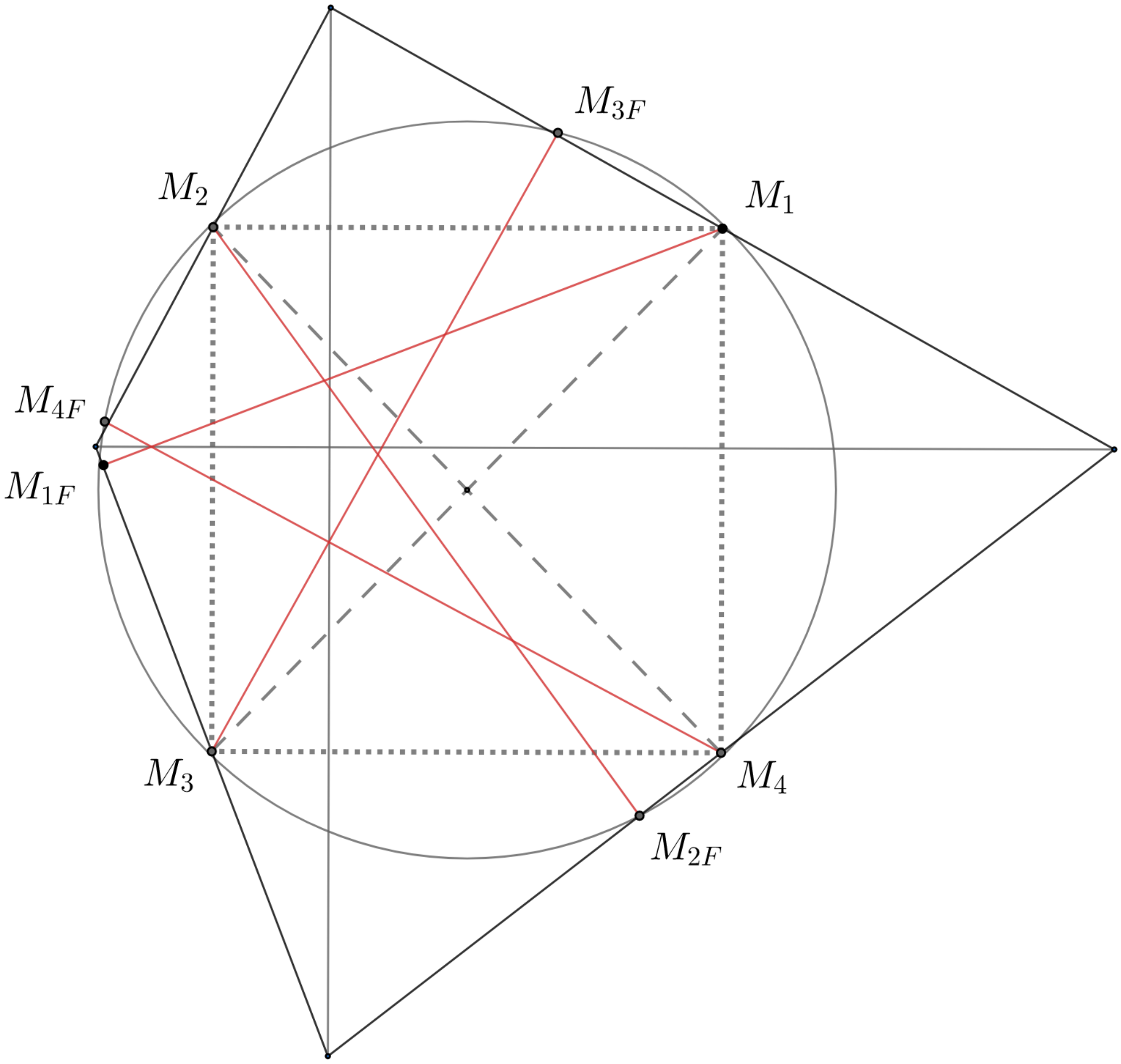
H *N* *G* *O*

3 **1** **2**

Here is a different derivation, due to:

L. Brand, 'The eight-point circle and the nine-point circle', *The American Mathematical Monthly* vol. 51, no. 84 (1944)

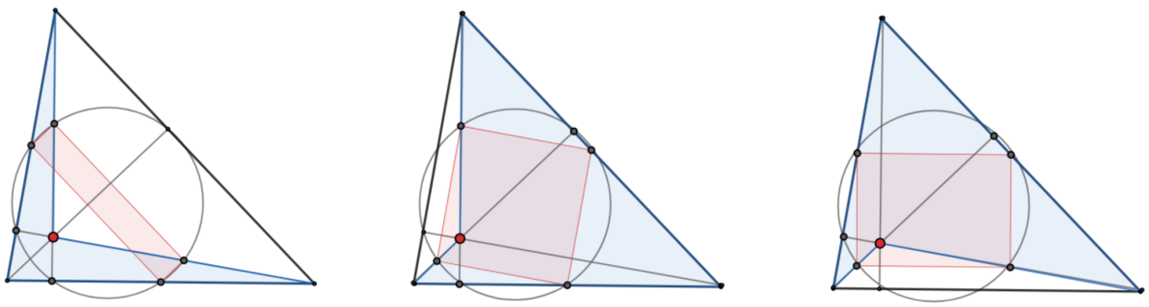
This one is instructive because it disguises a triangle as an orthodiagonal quadrilateral.



M_1, M_2, M_3, M_4 are the side midpoints.
Show that the midpoints of an orthodiagonal quadrilateral define a rectangle.

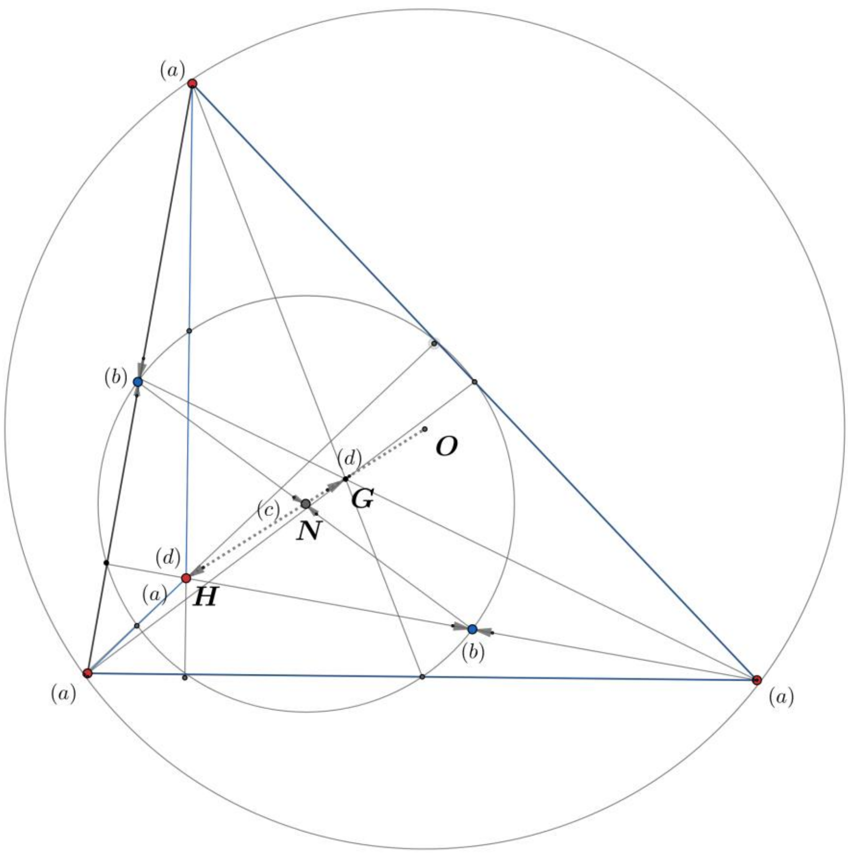
The diagonals of the rectangle are diameters of the circle shown.
$M_{1F}, M_{2F}, M_{3F}, M_{4F}$ are the altitudes from the side midpoints on to the opposite sides.
By the converse of Thales' theorem these points lie on the same circle.
For every orthodiagonal quadrilateral we therefore have an 8-point circle.

We now take rather special orthodiagonal quadrilaterals, namely the three we can dissect from a triangle with the addition of its orthocentre:



Trace the '8' points identified in the previous figure. The '8' is in inverted commas because two pairs have fused to make 6 in total. Moving from one figure to the next, you should find the figures retain three points in common. The circle in each case is therefore the same. Notice how the Euler points occur as the midpoints of the re-entrant sides. We end up with 9 concyclic points on the triangle: our 9-point circle.

In 'Mathematical Gems II' Ross Honsberger makes further use of the triangle-as-quadrilateral by allotting and distributing point masses among the vertices. There are four steps, labelled (a) to (d) in this figure:



(a) Allot a single mass to each vertex.
(b) Bring pairs to the midpoints of a pair of opposite sides.
(c) Bring all four to *their* midpoint, the centre of the nine-point circle, N.
(d) Since G is the centroid for the three point masses at the *triangle* vertices, we can maintain equilibrium about N by dividing the 4 masses into two lots: 3 to go to G and one to H. The distances $|NH|$ and $|NG|$ must stand in inverse ratio to this: 3 : 1.

We have therefore arrived by statics at a result we obtained above through transformation geometry.