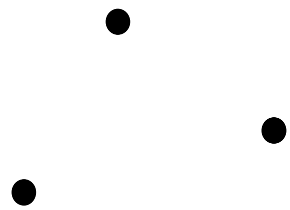
3.9.2 Motorway Networks

The Problem:

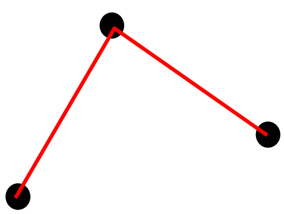
Suppose there are n points ('cities') drawn on a flat piece of paper. Our task is to draw a network of lines ('motorways') between these points in order to connect them together. Given any 2 points, there must be a way of getting from one to the other along the lines we have drawn.

e.g. For 3 cities (n = 3)

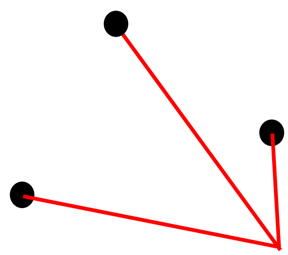


some possible networks are

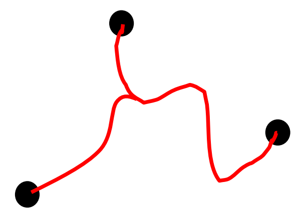
(i)



(ii)



(iii)



and there are many more…

The question is: ***Can we find the network whose lines have the shortest total length?***

In general this is quite a hard problem, but it is possible to discover some useful features of the solution.

Properties of the Solution:

Suppose we have found a network connecting our n cities which has the shortest total length possible. It must exhibit the following properties.

> *(i) The shortest network must comprise a finite number of 'nodes' – that is to say, the original n points, plus at most (n – 2) additional junctions (called "Steiner points"). These nodes are connected by straight lines in the network.*

This may be demonstrated as follows.

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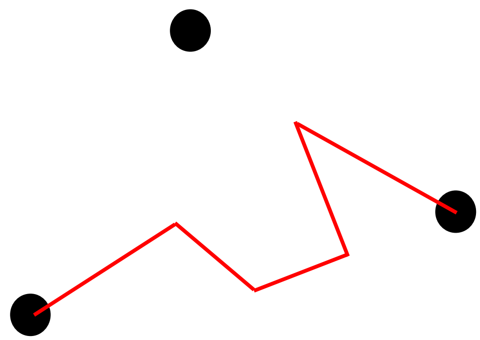
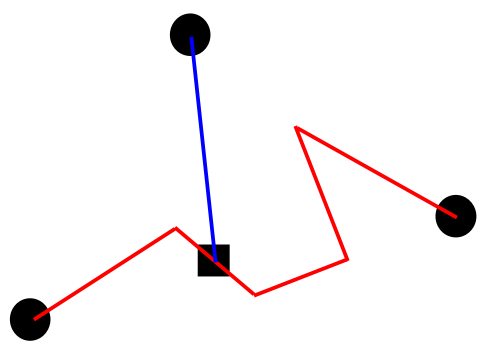
Pick any two points from the original ‘n’. There must be some line in the network (not necessarily straight) which joins the two points, as depicted below.

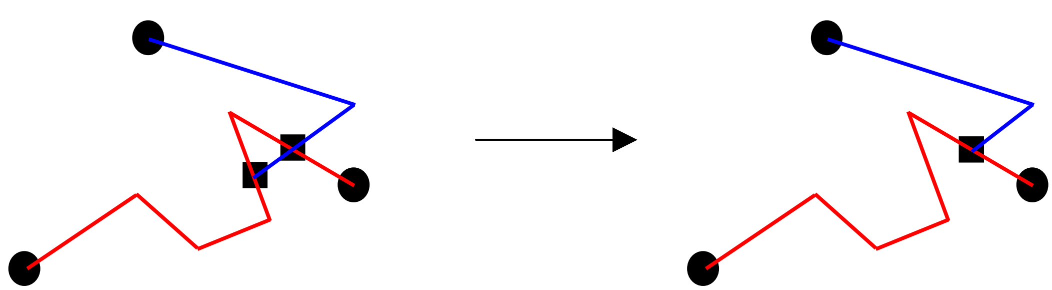


Now consider one of the other points. It must be connected to the original line (in red above) by some further line in the network. Otherwise, it would remain disconnected from the first two points.



This further line is shown in blue above. In the case depicted, we have introduced a new junction to the network (a Steiner point), which is here represented by a black square.

Note that we are free to choose our blue line such that it only intersects the previous network in **one** point. If, as is shown below, there were two or more intersections, we could simply restrict our attention to the first such junction and disregard the remainder of the line:



2 Intersections (unnecessary) Restrict to 1 Intersection

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This means that we are introducing **at most one** Steiner point by connecting our third city to the other two. (Of course, it is possible to do this without creating any Steiner points at all – if the blue line connects our third city directly to one of the first two cities, this will be the case.)

So now we have established a network of lines connecting 3 of the n cities together.

By precisely the same arguments as above, we may go on to consider a 4th city. This must be connected to the previous network of 3 cities by some line, and as before we can restrict our attention to a line which intersects the former network in just **one** point. Hence we arrive at a network of 4 cities, adding at most one Steiner point in the process.

Continuing in this way, we may pick out a set of lines from our hypothetical 'shortest network' which:

(a) Connects all 'n' cities together.
(b) Involves the addition of at most one Steiner point for each city beyond the first two.

Since this set of lines is part of the assumed 'shortest network', and since (a) tells us that it does the job of connecting all 'n' cities, it must BE the shortest network. Point (b) then tells us that the network contains at most (n – 2) Steiner points.

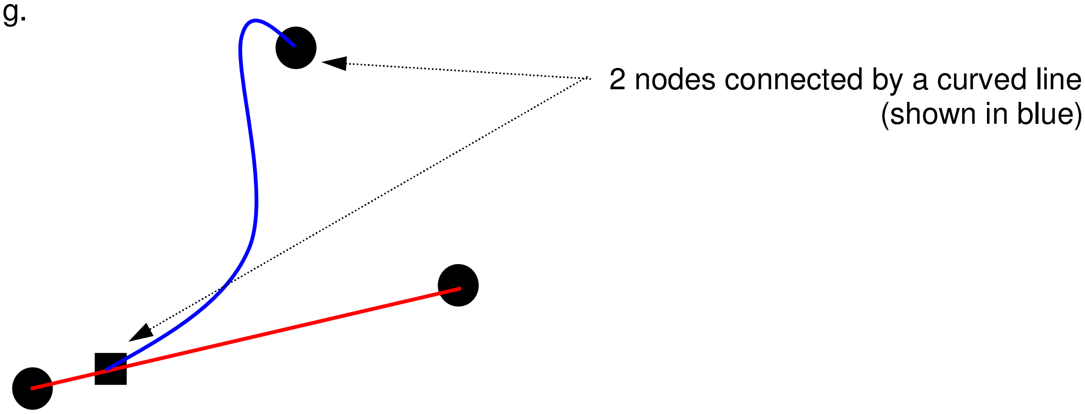
Summary

We have now shown that, given a collection of n cities, the shortest network connecting them together must have at most (2n – 2) nodes: the original n cities, plus no more than (n – 2) 'Steiner points'.

It remains to show that, if two nodes are directly connected by a line in the network, that line must be straight.

Suppose not.

e.g.



It is possible to construct a different network which connects the n cities, by replacing this curved line with a straight one.

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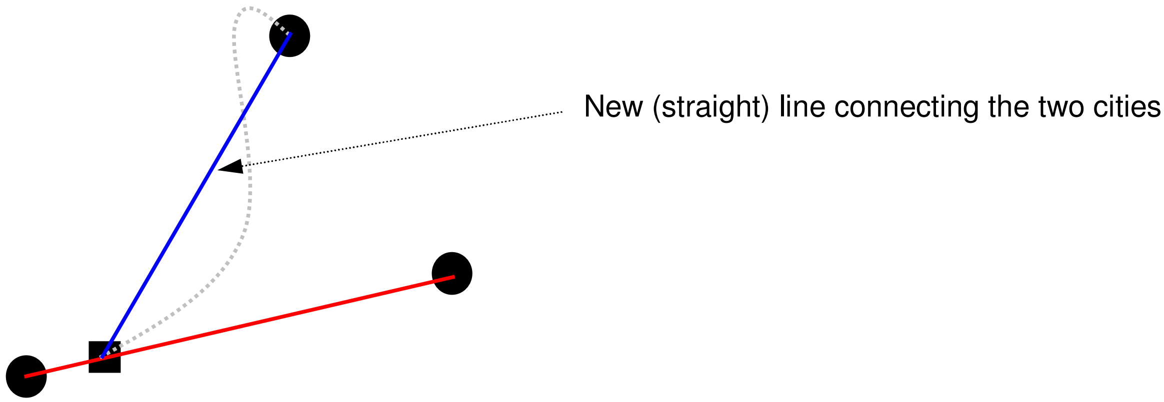
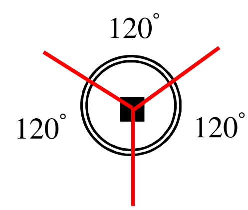


Since the shortest distance between two points is a straight line, the new network must be shorter than the previous one. But the previous one was our ‘shortest network’, so we have a contradiction!!

Consequently, a situation whereby there are curved/bent lines in the ‘shortest network’ must be impossible. This establishes the full boxed result from page 1.

(ii) Every additional node (Steiner point) must be a junction of precisely three straight lines, and these lines must intersect with adjacent angles of 120° (see diagram).

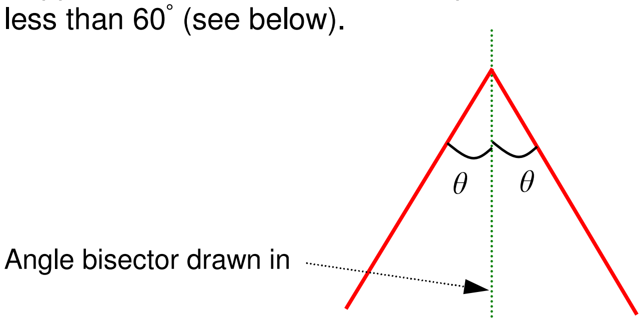


To prove this statement, it is useful to establish the following general result:

(iii) No two lines in the ‘shortest network’ can intersect at an angle less than 120°.

Proof

Suppose the network contains a pair of lines which intersect at an angle 2θ, where θ is less than 60° (see below).



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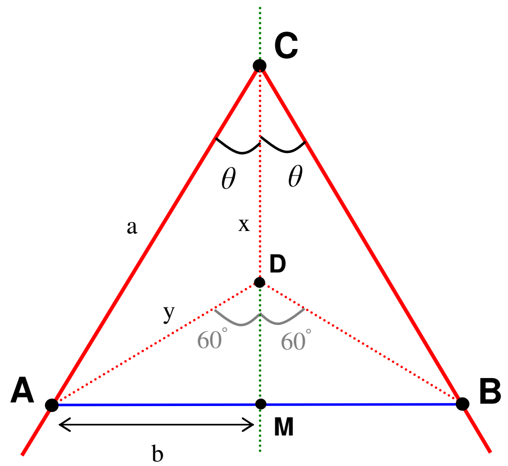
By marking off points A and B at equal distances from the vertex (labelled 'C' below), we can construct an isosceles triangle ABC:



Since $\theta < 60^\circ$, it is possible to find a point D on the angle bisector CM (and within the triangle itself!) such that

$$\angle ADM = \angle BDM = 60^\circ,$$

where M is the midpoint of AB.

Now let $AC\ (= BC) = $ 'a'
$AM\ (=BM) = $ 'b'
$AD\ (= BD) = $ 'y'
and $CD = $ 'x' as shown in the diagram.

Claim: $\boldsymbol{x + 2y < 2a}$

Proof of claim:

$$\begin{aligned} x &= CM - DM \\ &= a\cos\theta - b/(\tan 60^\circ) \\ &= a\cos\theta - b/(\sqrt{3}) \end{aligned}$$

$$\begin{aligned} y &= b/(\sin 60^\circ) \\ &= 2b/(\sqrt{3}) \end{aligned}$$

Therefore

$$\begin{aligned} x + 2y &= a\cos\theta - b/(\sqrt{3}) + 2.[2b/(\sqrt{3})] \\ &= a\cos\theta + 3b/(\sqrt{3}) \\ &= a\cos\theta + b\sqrt{3} \end{aligned}$$

But $b = a\sin\theta$, so

$$\begin{aligned} x + 2y &= a\cos\theta + (\sqrt{3}).a\sin\theta \\ &= 2a.(\tfrac{1}{2}\cos\theta + \tfrac{1}{2}(\sqrt{3})\sin\theta) \\ &= 2a.(\sin 30^\circ.\cos\theta + \cos 30^\circ.\sin\theta) \end{aligned}$$

Quoting the formula

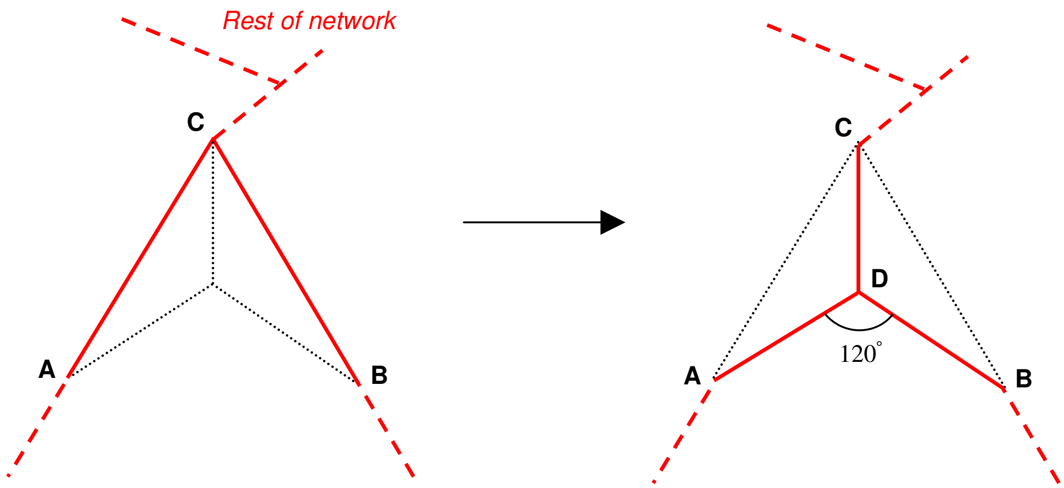
"$\boldsymbol{\sin(\alpha + \beta) = \sin\alpha.\cos\beta + \cos\alpha.\sin\beta}$", we obtain:

$$x + 2y = 2a.\sin(30^\circ + \theta)$$

As $0 < \theta < 60^\circ$, we have $30^\circ < (30^\circ + \theta) < 90^\circ$
and hence $\sin(30^\circ + \theta) < \sin(90^\circ) = 1$

It follows that $x + 2y < 2a$, as claimed. □

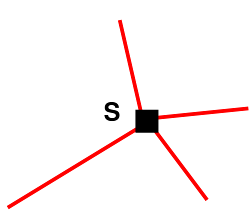
Now suppose we take the original network (presumed to be the 'shortest' network between our n points) and replace the lines AC and BC by the three lines AD, BD and CD from the previous diagram:



The new network does exactly the same job of connecting points together, but has a shorter total length than the original one (because AD + BD + CD < AC + BC, as proven above). This contradicts our assumption that the original network was the shortest possible one.

We may deduce that the genuine shortest network, whatever it may be, possesses no pair of lines intersecting at an angle less than 120°. This establishes statement (iii) from page 4.

It is only a small jump from here to statement (ii). For suppose we have a Steiner point S in the shortest network:

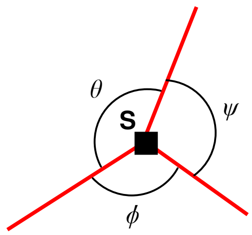


If more than 3 lines come together in the point S, then there must be a pair amongst these lines which is separated by an angle less than 120° (contradicting our earlier result).

Consequently, this is not possible.

If S is the junction of precisely 3 lines in the network, then each pair of adjacent lines must be separated by an angle greater than or equal to 120° [from statement (iii)].

However, the sum of the three angles must be 360˚… this means that all 3 angles must be precisely 120˚, since if one were larger than this at least one of the others would have to be smaller to compensate.



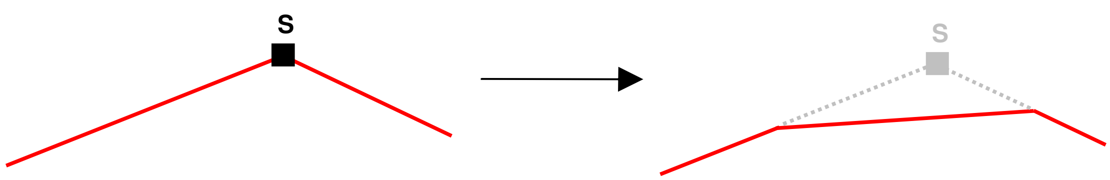
$$\theta, \phi, \psi \geq 120^\circ \quad \textbf{and} \quad \theta + \phi + \psi = 360^\circ$$

$$\Rightarrow \quad \theta = 360^\circ - \phi - \psi$$
$$\leq 360^\circ - 120^\circ - 120^\circ = 120^\circ$$

$$\text{So} \quad \theta \geq 120^\circ \quad \text{and} \quad \theta \leq 120^\circ,$$

$$\text{and hence} \quad \boldsymbol{\theta = 120^\circ}. \quad (\text{Similarly for } \phi \text{ and } \psi)$$

The only remaining case to consider is when S is the junction of exactly 2 lines. However, no such Steiner point can exist in our shortest network: if it did, then the network could be shortened in the way depicted below (which would contradict its status as 'shortest'!).



Steiner point connecting 2 lines

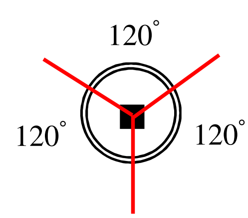
Shorter alternative!

We have now demonstrated that each Steiner point in the shortest network must be a junction of precisely 3 straight lines, and that these lines must be separated by adjacent angles of 120˚. Hence statement (ii) from page 4.

Overall Summary:

(i) The shortest network must comprise a finite number of 'nodes' – that is to say, the original n points, plus at most (n – 2) additional junctions (called "Steiner points"). These nodes are connected by straight lines in the network.

(ii) Every additional node (Steiner point) must be a junction of precisely three straight lines, and these lines must intersect with adjacent angles of 120˚ (see diagram).



(iii) No two lines in the 'shortest network' can intersect at an angle less than 120˚.